α_s from a nonperturbative determination of the QCD Λ-parameter

Patrick Fritzsch

<patrick.fritzsch@cern.ch>

on behalf of the ALPHA Collaboration

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Quantum Chromodynamics

In Euclidean space with gauge group $\mathrm{SU}(3)$ and N_f quark flavours:

$$\mathcal{L}^{\mathrm{QCD}} = -\frac{1}{2g^2}\mathrm{Tr}[F_{\mu\nu}F_{\mu\nu}] + \sum_{i=1}^{N_\mathrm{f}} \overline{\psi}_i[\gamma_\mu D_\mu + m_i]\psi_i$$

describes a plethora of strongly interacting processes

- gauge invariant
- $N_\mathrm{f}+1$ free parameters $\left\{ \begin{array}{ll} \text{strong coupling} & g^2 \\ \text{quark masses} & m_i, i=1,\ldots,N_\mathrm{f} \end{array} \right\}$ require physical input

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- regularize & renormalize $\rightsquigarrow \bar{g}(\mu), \bar{m}_i(\mu)$
- massless Renormalization Group Eq. (RGE):

mass anomalous dimension

$$\tau(\bar{g}) \equiv \frac{\mu}{\bar{m}_i(\mu)}\frac{\partial \bar{m}_i(\mu)}{\partial \mu} ,$$

β-function

$$\beta(\bar{g}) \equiv \mu\frac{\partial \bar{g}(\mu)}{\partial \mu}$$

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see EPJ C78 (2018) 387[5, 6]

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hadronic input $m_\pi, f_\pi, \ldots$

mass anomalous dimension

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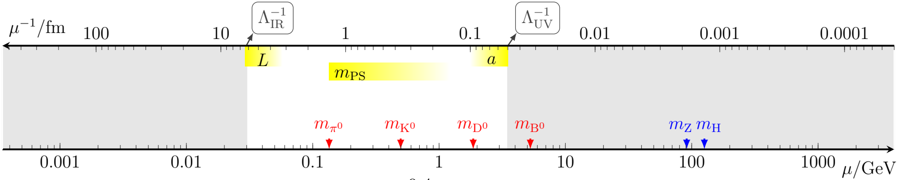
- **Challenge:** *precise & accurate (high-quality) determination of α_s from 1st principles (Lattice QCD)*
- **Pitfall:** $\alpha_s(\mu)$ traditionally quoted at $\mu = m_Z$ in $\overline{\mathrm{MS}}$ scheme



$$\beta(\bar{g}) \equiv Q \frac{\partial}{\partial Q} \bar{g}(Q)$$

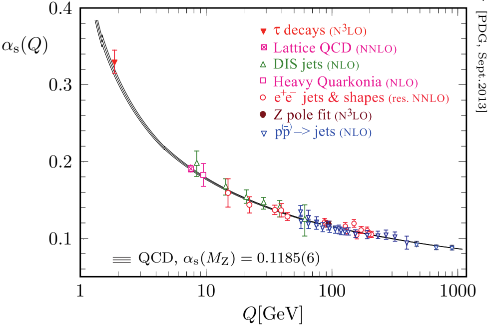
$$\Updownarrow$$

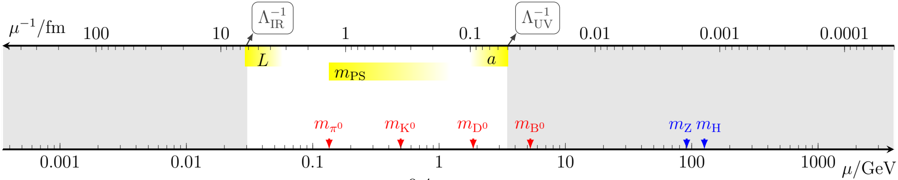
$$\ln\left[\frac{\mu}{\mu_0}\right] = \int_{\bar{g}(\mu_0)}^{\bar{g}(\mu)} \frac{\mathrm{d}g}{\beta(g)}$$

$$\Updownarrow$$

$$\Lambda^{(N_{\rm f})} = \lim_{\mu\to\infty} \mu \left[b_0 \bar{g}^2(\mu)\right]^{\frac{-b_1}{2b_0^2}}$$

RGEs valid & exact beyond PT





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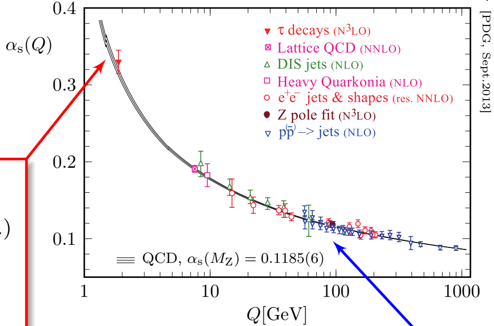


Asymptotic series at large α_s?

$$\beta(g) \overset{\bar{g}\to 0}{\sim} -\bar{g}^3(b_0 + b_1\bar{g}^2 + b_2\bar{g}^4 + \ldots)$$

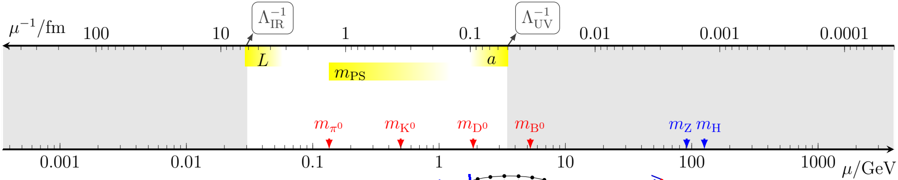
a) series truncation
b) non-perturbative effects
(instantons, renormalons, you name it)

$\Downarrow$

Bias when lattice obs. matched to PT.

Perform matching to PT at small α_s!

Running coupling and Lattice QCD



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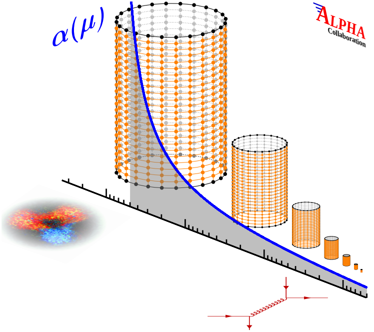

Solve RGEs non-perturbatively

- massless finite-volume scheme:[7]
- special purpose observables:[8, 9]
- recursive finite-size scaling:[10]

$m = 0,\ \mu = 1/L$

$\bar g^2_{\rm SF}(L),\ \bar g^2_{\rm GF}(L)$

$$\bar g^2(2L) \equiv \sigma\left(\bar g^2(L)\right) = \lim_{a\to 0} \Sigma\left(\bar g^2(L), a/L\right)$$

$\Rightarrow$ non-perturbative continuum $\beta(\bar g)$ for given scheme

$\mu_{\rm max}/\mu_{\rm min} = {\rm O}(100)$

Our results for $N_\mathrm{f} = 3$[11]

$$\Lambda \equiv \mu \left[b_0 \bar{g}^2(\mu)\right]^{-b_1/(2b_0^2)} \mathrm{e}^{-1/(2b_0\bar{g}^2(\mu))} \exp\left\{-\int_0^{\bar{g}(\mu)} \mathrm{d}g \left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\}$$

- *exact equation* $\forall \mu$ (or $\bar{g}$) in massless scheme for given N_f and non-perturbative $\beta(g)$
- *trivial scheme dependence:* $\Lambda_a/\Lambda_b = \exp(c_{ab}/2b_0)$ is 1-loop exact, e.g., $\Lambda_{\overline{\mathrm{MS}}}/\Lambda_{\mathrm{SF}} = \mathrm{const}$

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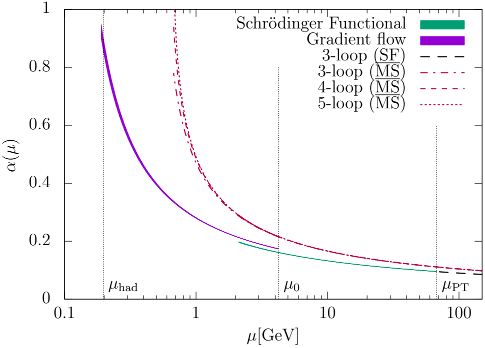

- PDG input enters $f_\mathrm{had}^\mathrm{PDG}$

  $m_\pi, m_\mathrm{K}, f_\pi, f_\mathrm{K}$

- $\bar{g}^2_\mathrm{GF}(\mu_\mathrm{had}) \equiv 11.31$

  $\mu_0/\mu_\mathrm{had} = 21.86(42)$

- switch: $\bar{g}^2_\mathrm{GF}(2\mu_0) = 2.6723(64)$
- $\bar{g}^2_\mathrm{SF}(\mu_0) = 2.012$

  $\Lambda^{(3)}_{\overline{\mathrm{MS}}}/\mu_0 = 0.0791(21)$

- $\mu_\mathrm{PT}/\mu_\mathrm{had} = 349.7(6.8)$

$$\underbrace{f_\mathrm{had}^\mathrm{PDG} \times \frac{\mu_\mathrm{had}}{f_\mathrm{had}}}_{\text{LV scale setting}} \times \underbrace{\frac{2\mu_0}{\mu_\mathrm{had}}}_{\text{GF running}} \times \underbrace{\frac{\mu_0}{2\mu_0}}_{\text{scheme change}} \times \underbrace{\frac{\mu_\mathrm{PT}}{\mu_0}}_{\text{SF running}} \times \underbrace{\frac{\Lambda^{(3)}_\mathrm{SF}}{\mu_\mathrm{PT}}}_{\text{PT@65GeV}} \times \underbrace{\frac{\Lambda^{(3)}_{\overline{\mathrm{MS}}}}{\Lambda^{(3)}_\mathrm{SF}}}_{\text{exact}} \equiv \Lambda^{(3)}_{\overline{\mathrm{MS}}}$$

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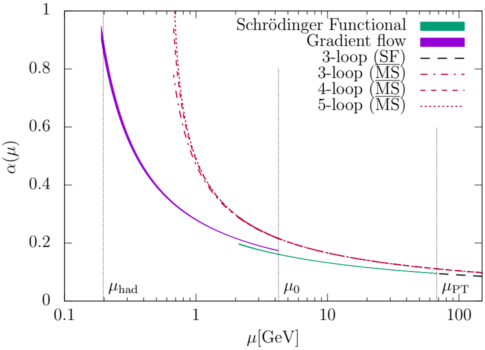

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$$\Lambda^{(3)}_{\overline{\mathrm{MS}}} = 341(12)\,\mathrm{MeV}$$

$\Lambda^{(3)}_{\overline{\rm MS}} \to \alpha_s^{(5)}(m_Z)$ via perturbative decoupling

Decoupling relation

$$\bar{g}^{N_f}(\mu) = \bar{g}^{N_f+1}(\mu) \times \xi\left(g^{N_f}(\mu), \overline{m}_h/\mu\right) \quad + \quad O(\overline{m}_h^{-2})$$

or equivalently relation for $\Lambda^{(N_f)}/\Lambda^{(N_f+1)}$

- requires further PDG input ($\overline{\rm MS}$ scheme)

  $\overline{m}_c(\overline{m}_c)$ $\quad\quad$ $\overline{m}_b(\overline{m}_b)$

- $O(\overline{m}_h^{-2})$ already very small[12, 13] for $h = c$
- ξ known in PT to 4 loops[14, 15]
- for decoupling perturbation theory looks surprisingly well-behaved already at $\mu = \overline{m}_c$
- Future: include charm non-perturbatively

n (loops)	$\alpha^{(N_f=5)}_{\overline{\rm MS}}$	$\alpha_n - \alpha_{n-1}$
1	0.11699	
2	0.11827	0.00128
3	0.11846	0.00019
4	0.11852	0.00006

$\Delta\alpha = \alpha_4 - \alpha_2 \approx 0.00025$

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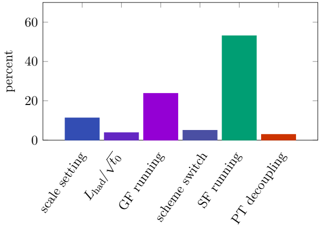
$$\alpha_s^{(5)}(m_Z) = 0.11852(80)(25)\ \text{MeV}$$

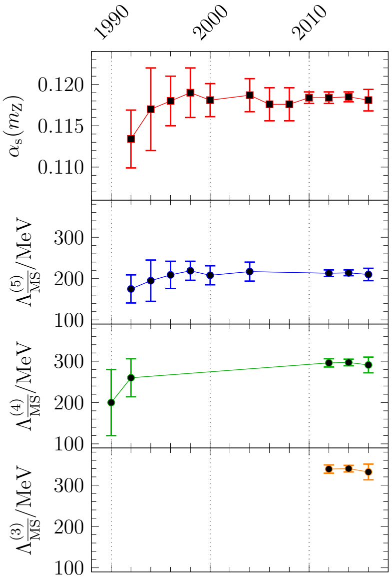
PDG-16:[16] $\alpha_s^{(5)}(m_Z) = 0.1174(16)$ w/o lattice

FLAG-16:[17] $\alpha_s^{(5)}(m_Z) = 0.1182(12)$

Contribution to relative error squared



Historic $\alpha_s(m_Z)$ averages



20 years of PDG data history

Note:

- recent *increase of uncertainty*
- quoting $\alpha_s^{\overline{MS}}$ at $\mu = m_Z$ is a *convention*

PDG-2016 values:

$$\alpha_s(m_Z) = 0.1181(11) \qquad \sim 0.9\%$$

$$\Lambda^{(5)}_{\overline{MS}} = 210(14)\ \text{MeV} \qquad \sim 6.7\%$$

$$\Lambda^{(4)}_{\overline{MS}} = 292(16)\ \text{MeV} \qquad \sim 5.5\%$$

$$\Lambda^{(3)}_{\overline{MS}} = 332(17)\ \text{MeV} \qquad \sim 5.1\%$$

Our results:

$$\Lambda^{(3)}_{\overline{MS}} = 341(12)\ \text{MeV} \qquad \sim 3.5\%$$

$$\alpha_s(m_Z) = 0.11852(84) \qquad \sim 0.7\%$$

Experimental support for QCD

RG running of α_s, past and present

tremendous progress over the years

RG scale evolution consistent with data

1989 Altarelli[18]

2016 CMS[19]

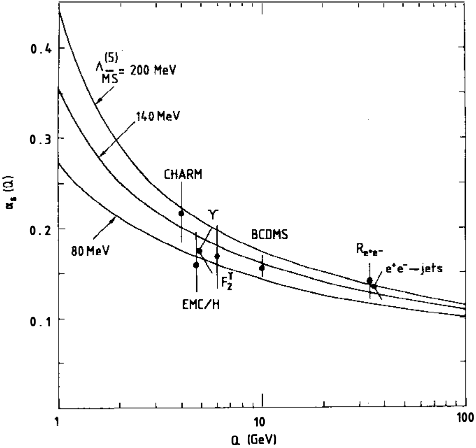
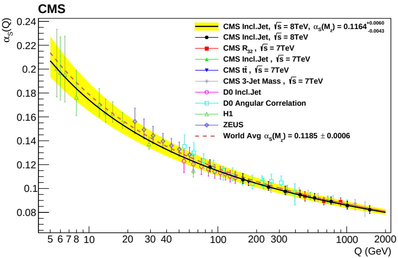


Figure 7 A summary of the determinations of the running coupling constant α_s discussed in the text. The curves for $\Lambda^{(5)}_{\overline{MS}} = 140 \pm 60$ MeV are obtained following the matching procedure at the b threshold explained in Equations 14–19 (with $a \approx 1$).

$$\alpha_s^{(6)}(1.508\ \mathrm{TeV}) = 0.0840(35)$$



ALPHA Collaboration : $\alpha_s^{(6)}(1.508\ \mathrm{TeV}) = 0.0852(4)$

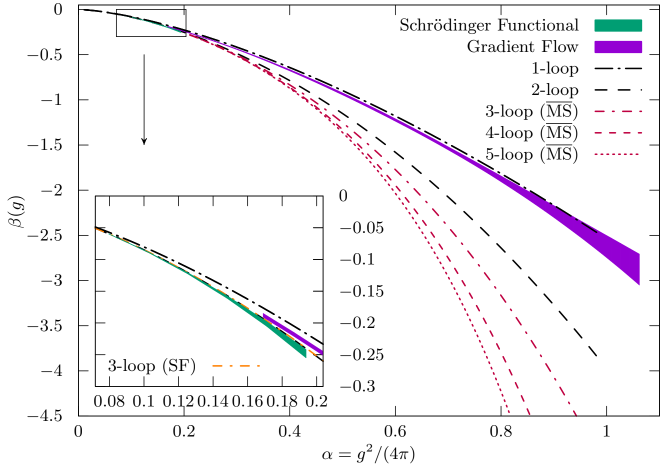
THANK YOU FOR YOUR ATTENTION!

And many thanks to my collaborators:

- M. Bruno
- M. Dalla Brida
- T. Korzec
- A. Ramos
- S. Schaefer
- H. Simma
- S. Sint
- R. Sommer

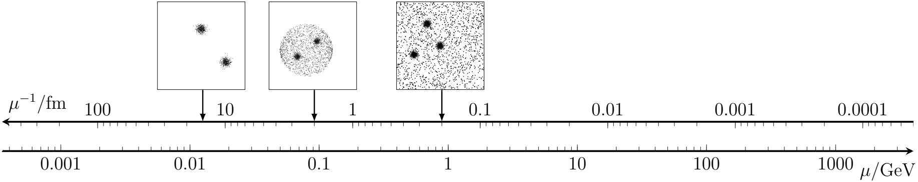
Backup slides

Non-perturbative β functions



QCD in a box:

Bjorken[20]



$\leftsquigarrow$ confinement region

asymptotic freedom ($\mu \to \infty$)

QCD at different length scales:

- 2 hadrons in a box of size $L = 15\,\mathrm{fm}$
- a $q\overline{q}$ meson in a $2\,\mathrm{fm}$ box
- interacting quarks and gluons at $L = 0.2\,\mathrm{fm}$

The Femtouniverse ($L \ll 0.5\,\mathrm{fm}$):

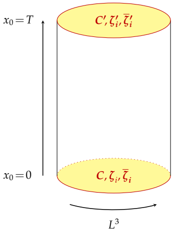
- in principle, QCD phenomena can be probed at any scale
- a single experiment cannot achieve that
- ideal playground for lattice QCD as a non-perturbative toolbox at all scales
  (boundary conditions very important at intermediate or small box sizes)

The Schrödinger functional coupling

- Euclidean partition function

$$\mathcal{Z} \equiv \int_{T\times L^3} \mathcal{D}\left[U,\overline{\psi},\psi\right] \mathrm{e}^{-S[U,\overline{\psi},\psi]} = \langle 0|\,\mathrm{e}^{-TH}P|0\rangle$$

  with *periodic BC in* L^3
  and *Dirichlet BC in* T (breaking translational inv. in time)



- renormalization scale $\mu \propto L^{-1}$ (for step-scaling)
- mass-independent scheme, . . .

Abelian boundary fields: $C_k = \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_3 \end{pmatrix}$; $C'_k = \begin{pmatrix} \phi'_1 & 0 & 0 \\ 0 & \phi'_2 & 0 \\ 0 & 0 & \phi'_3 \end{pmatrix}$

SF coupling

defined as variation of effective action $\Gamma = -\ln \mathcal{Z}[C, C']$,

$$\left.\frac{\partial\Gamma}{\partial\eta}\right|_{\eta=0} = \frac{\mathrm{const}}{\bar{g}^2_{\mathrm{SF}}(L)}$$

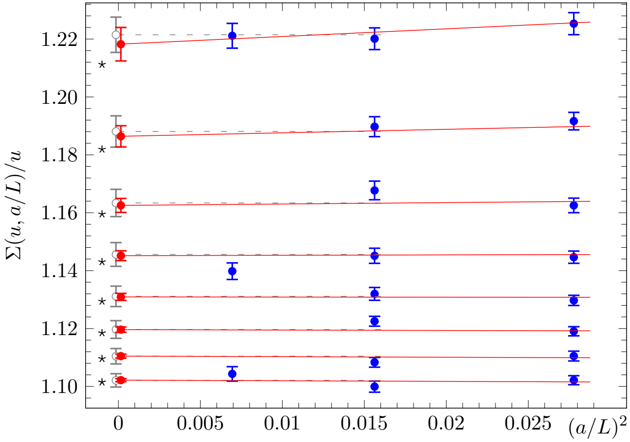
for non-vanishing boundary gauge fields $C_k \neq 0 \neq C'_k$

Non-perturbative running at high energies

Continuum extrapolation of SSF Σ_{SF}[1]

Example for global fit ansatz: (4 parms., 19 pts., $\chi^2/N_{\mathrm{dof}} \approx 1$)

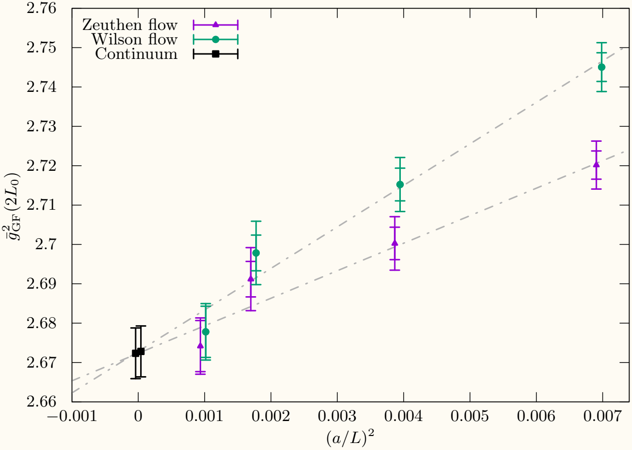
$$\Sigma(u, a/L) = u + s_0 u^2 + s_1 u^3 + c_1 u^4 + c_2 u^5 + \rho_1 u^4 (\tfrac{a}{L}) + \rho_2 u^5 (\tfrac{a}{L})^2$$



s_0, s_1 fixed to perturbative values: $s_0 = 2b_0 \ln(s)$, $s_1 = s_0^2 + 2b_1 \ln(s)$, $s = 2$

Matching at the scheme switching scale $L_{\rm swi}$

$$\bar{g}^2_{\rm SF}(L_{\rm swi}) = 2.0120 \quad \Rightarrow \quad \bar{g}^2_{\rm GF}(2L_{\rm swi}) = 2.6723(64)$$

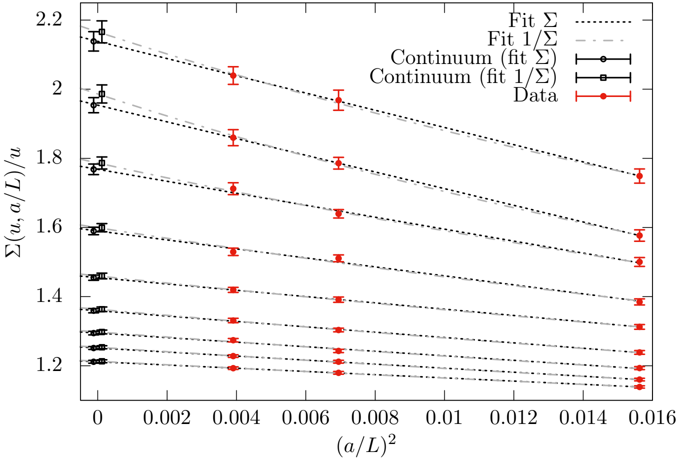


- 0.2% total uncertainty (continuum)[2]

$2L_{\rm swi}/a \in \{12, 16, 24, 32\}$

Non-perturbative running at low energies

Taking the continuum limit[2]



- sizeable discretization effects $\rightarrow$ requires **careful** extrapolations

$$2L/a \in \{16, 24, 32\}$$

- nonetheless, continuum results are very **precise**!

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