KUNGL. VETENSKAPS-AKADEMIEN

THE ROYAL SWEDISH ACADEMY OF SCIENCES

The Royal Swedish Academy of Sciences has decided to award the Nobel Prize in Physics for 2013 to **François Englert** and **Peter W. Higgs** "for the theoretical discovery of a mechanism that contributes to our understanding of the origin of mass of subatomic particles, and which recently was confirmed through the discovery of the predicted fundamental particle, by the ATLAS and CMS experiments at CERN's Large Hadron Collider".

The Nobel Prize 2013 **in Physics**

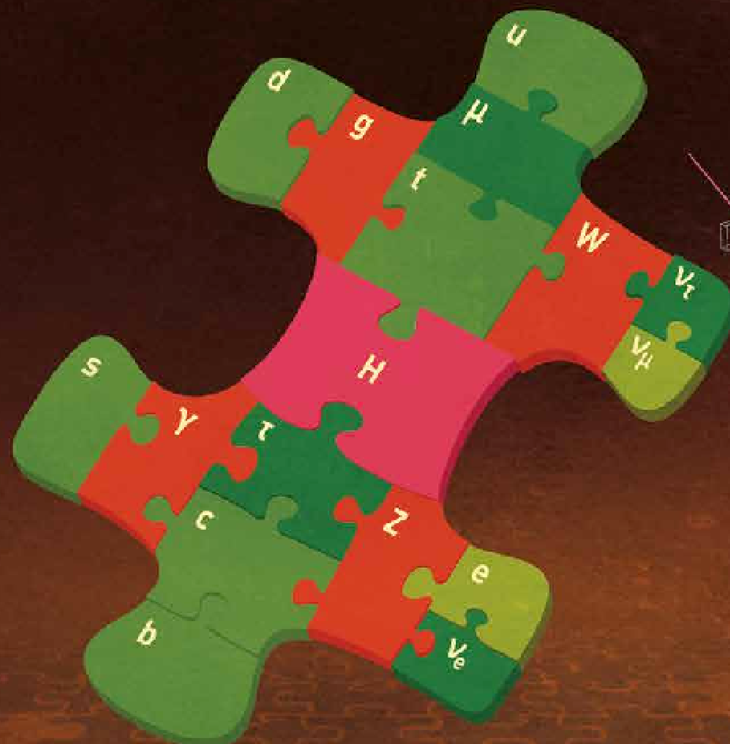
Here, at last!

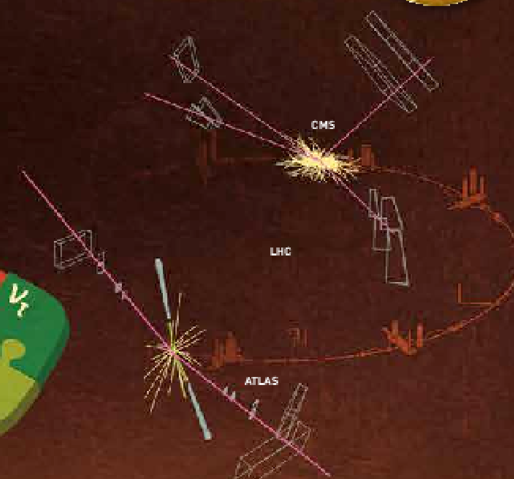
François Englert and Peter W. Higgs are jointly awarded the Nobel Prize in Physics 2013 for the theory of how particles acquire mass. In 1964, they proposed the theory independently of each other (Englert did so together with his now-deceased colleague Robert Brout). In 2012, their ideas were confirmed by the discovery of a so-called Higgs particle, at the CERN laboratory outside Geneva in Switzerland.

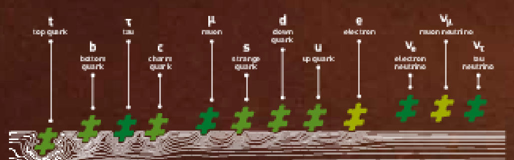
The awarded mechanism is a central part of the Standard Model of particle physics that describes how the world is constructed. According to the Standard Model, everything – from flowers and people to stars and planets – consists of just a few building blocks: *matter particles* which are governed by forces mediated by *force particles*. And the entire Standard Model also rests on the existence of a special kind of particle: *the Higgs particle*.

The Higgs particle is a vibration of an invisible field that fills up all space. Even when our universe seems empty, this field is there. Had it not been there, nothing of what we know would exist because particles acquire mass only in contact with the Higgs field. Englert and Higgs proposed the existence of the field on purely mathematical grounds, and the only way to discover it was to find the Higgs particle.

The Nobel Laureates probably did not imagine that they would get to see the theory confirmed in their lifetimes. To do so required an enormous effort by physicists from all over the world. Almost half a century after the proposal was made, on July 4, 2012, the theoretical prediction could celebrate its biggest triumph, when the discovery of the Higgs particle was announced.







ATLAS

In the collision, a short-lived Higgs particle is created, which decays into two muons (tracks in red) and two electrons (tracks in green).

CMS

A short-lived Higgs particle is created in the collision and decays into four muons (tracks in red).

The Particle Collider LHC

Protons – hydrogen nuclei – travel at almost the speed of light in opposite directions inside the circular tunnel, 27 kilometres long. The LHC (Large Hadron Collider) is the largest and most complex machine ever constructed by humans. In order to find a trace of the Higgs particle, two huge detectors, ATLAS and CMS, are capable of seeing the protons collide over and over again, 40 million times a second.

↑

The Field

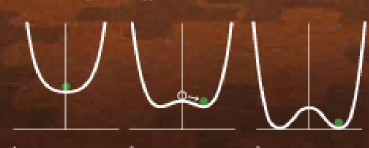
Matter particles acquire mass in contact with the invisible field that fills the whole universe. Particles that are not affected by the Higgs field do not acquire mass, those that interact weakly become light, and those that interact strongly become heavy. For example, electrons acquire mass from the field, and if it suddenly disappeared, all matter would collapse as the suddenly massless electrons dispersed at the speed of light. The weak force carriers, W and Z particles, get their masses directly through the Higgs mechanism, while the origin of the neutrino masses still remains unclear.

→

Broken Symmetry

The Higgs mechanism relies on the concept of spontaneous symmetry breaking. Our universe was probably born symmetrical (1), with a zero value for the Higgs field in the lowest energy state – the vacuum. But less than one billionth of a second after the Big Bang, the symmetry was broken spontaneously as the lowest energy state moved away (2) from the symmetrical zero-point. Since then, the value of the Higgs field in the vacuum state has been non-zero (3).

Potential energy of the Higgs field



The Puzzle

The Higgs particle (H) was the last missing piece in the Standard Model puzzle. But the Standard Model is not the final piece in the cosmic puzzle. One of the reasons for this is that the Standard Model only describes visible matter, accounting for one sixth of all matter in the universe. To find the rest – the mysterious so-called dark matter – is one of the reasons why scientists continue to chase unknown particles at CERN.

← ←

François Englert

Belgian citizen. Born 1932 in Etterbeek, Belgium. Professor emeritus at Université Libre de Bruxelles, Brussels, Belgium.

←

Peter W. Higgs

British citizen. Born 1929 in Newcastle upon Tyne, United Kingdom. Professor emeritus at University of Edinburgh, United Kingdom.

FURTHER READING!

What's the Matter?!

Karlsruhe Institute of Technology

Gravitational Force

Reach : ∞
Strength : 10^{-41}
Charge : mass

Weak Force

Reach : $10^{-17}\mathrm{m}$
Strength : 10^{-14}
Charge : weak isospin

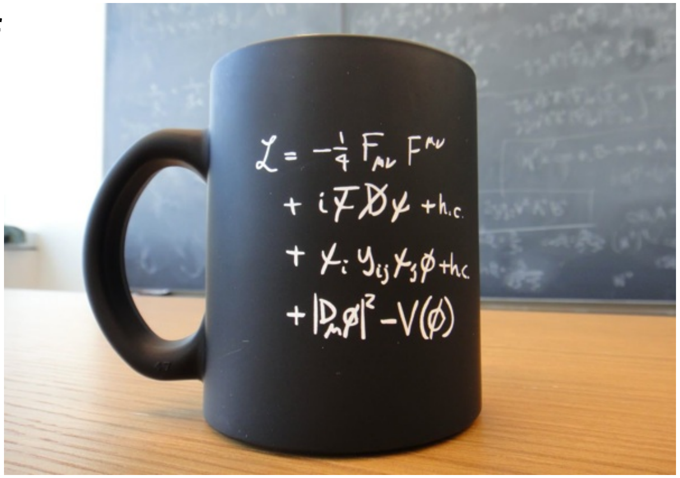
Electromagnetic Force

Reach : ∞
Strength : $1/137$
Charge : electric charge

Strong Force

Reach : $10^{-15}\mathrm{m}$
Strength : $O(1)$
Charge : color

"Simple" principle of *Gauge Symmetries*!



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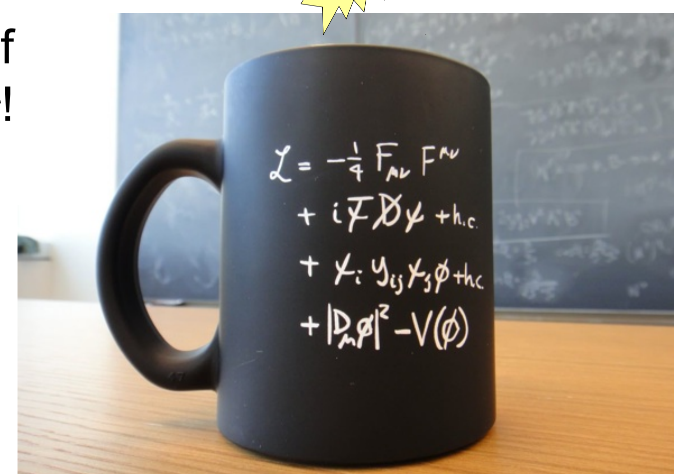
Grinch



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Higgs Mechanism

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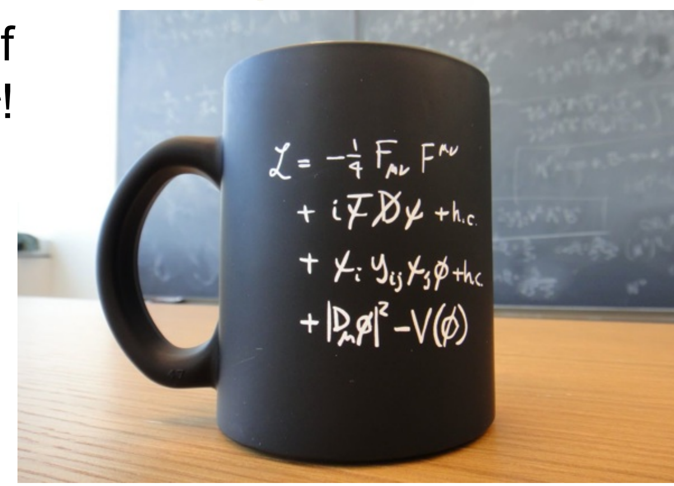
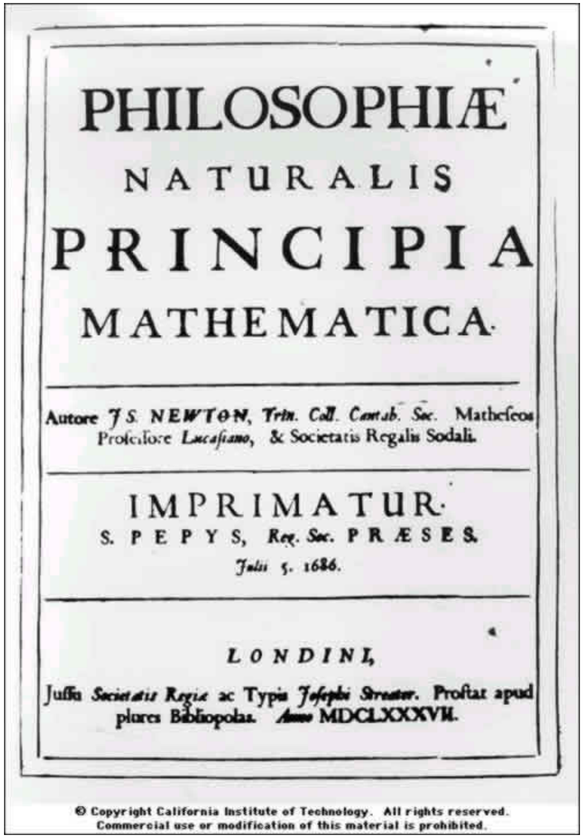


Mass $\neq$ Mass


PHILOSOPHIÆ
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MATHEMATICA.

Autore J S. NEWTON, Trin. Coll. Cantab. Soc. Matheſeos Profeſſore Lucaſiano, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, Reg. Soc. PRÆSES.
Julii 5. 1686.

LONDINI,
Juſſu Societatis Regiæ ac Typis Joſephi Streater. Proſtat apud plures Bibliopolas. Anno MDCLXXXVII.

Newton's law of gravitation:

$$m \cdot \vec{a} = G\frac{m \cdot M}{\vec{r}^2} \cdot \frac{\vec{r}}{|\vec{r}|}$$

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0$$

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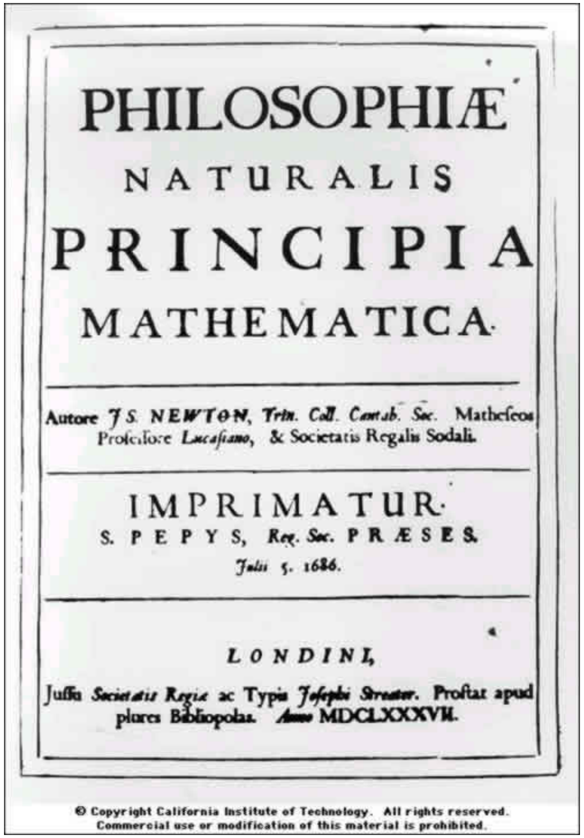
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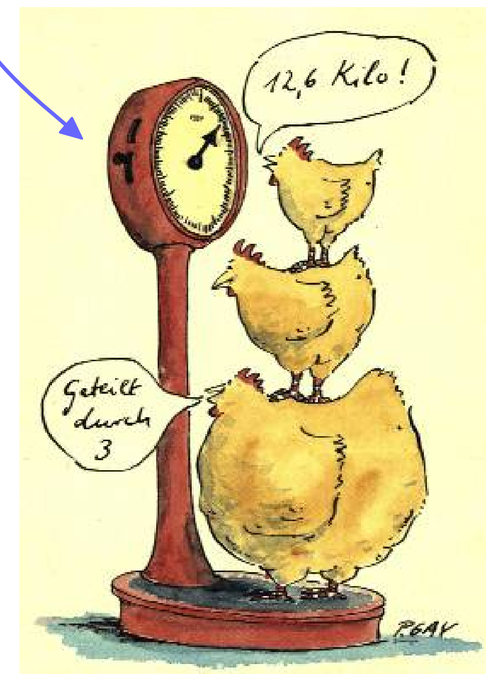

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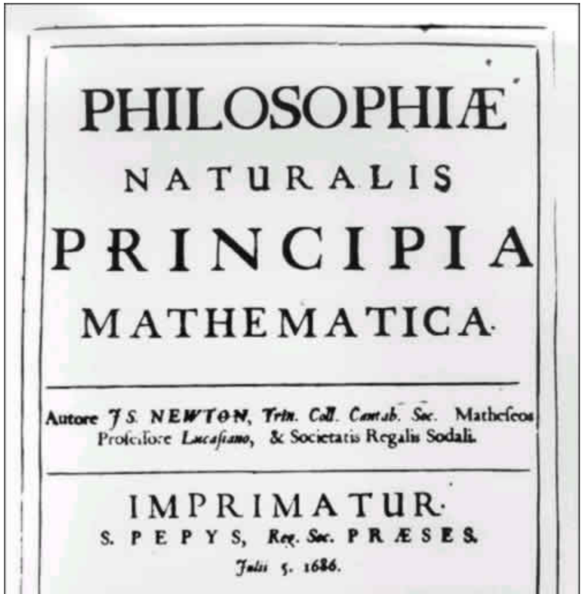
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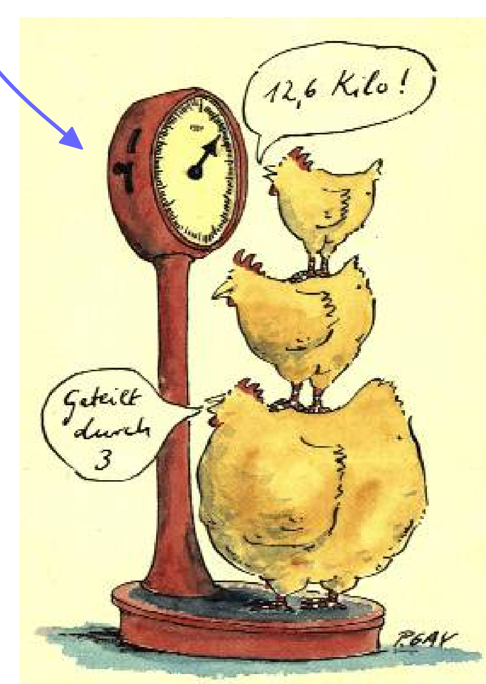
mass of inertia



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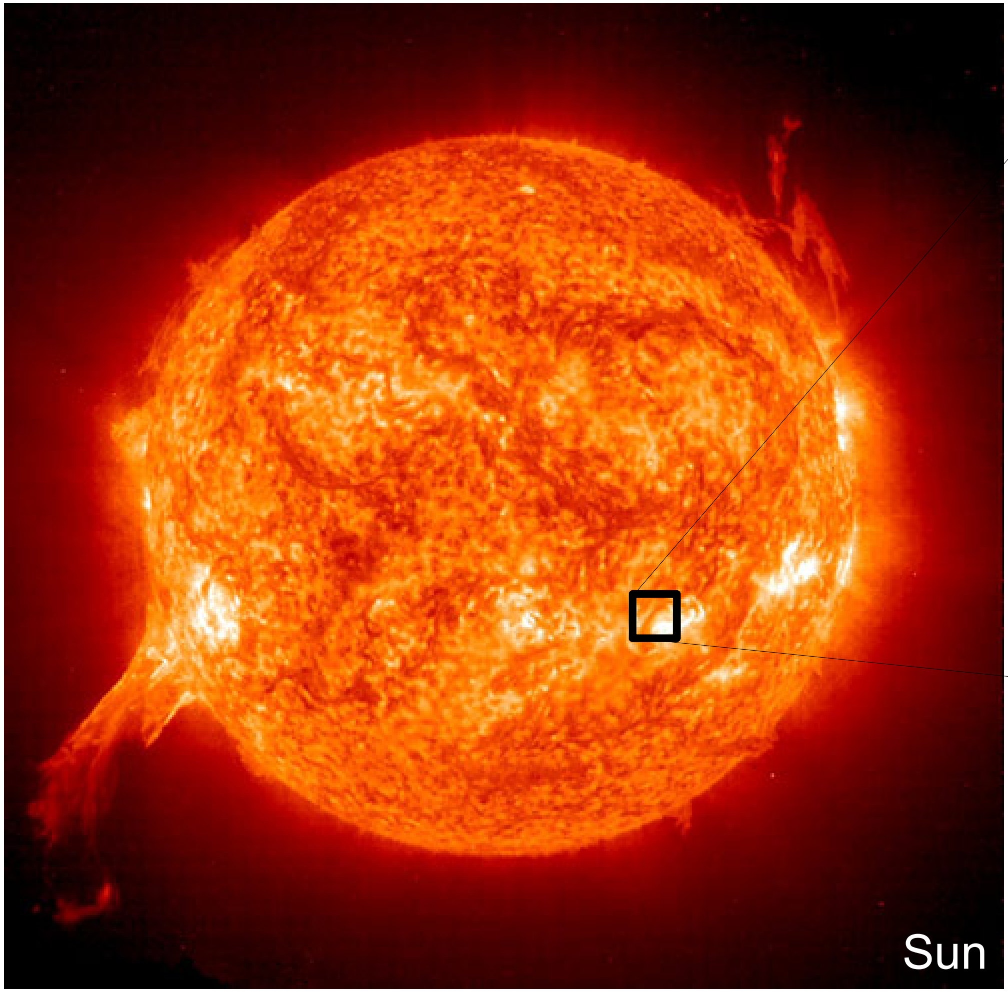
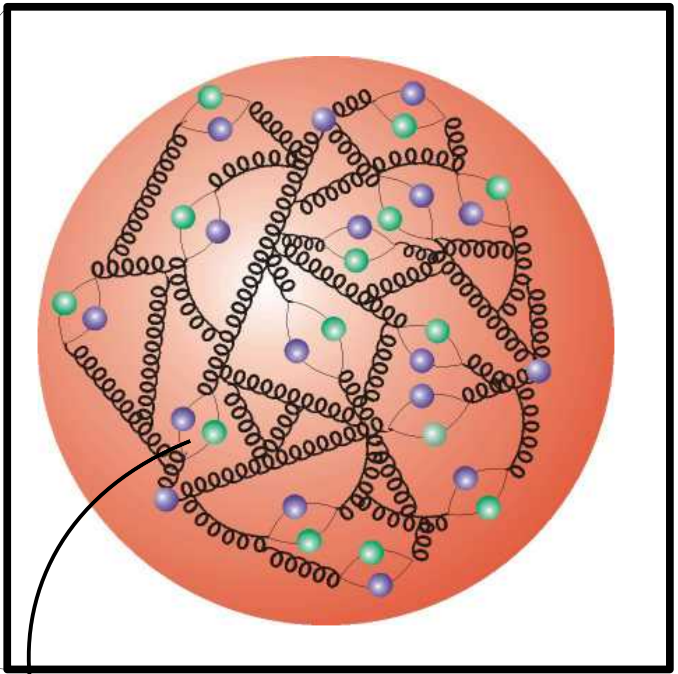




$$m_p = 938.3 \text{ MeV}$$

$$m_u = 2 - 3 \text{ MeV}$$

$$m_u \approx {}^{1}\!/_{500} \cdot m_p$$

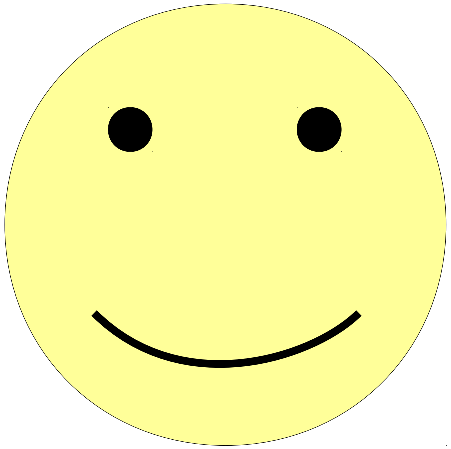
So, what's the importance then of m ?!?

Without m ...

- … no Newtonian Laws.

Without m ...

- … no Newtonian Laws.
- … everybody would move at the speed of light.



Without m ...

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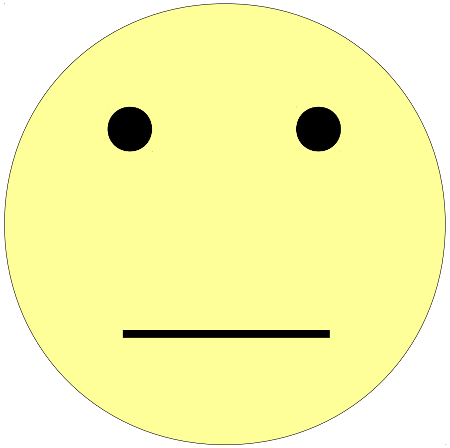

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Without m ...

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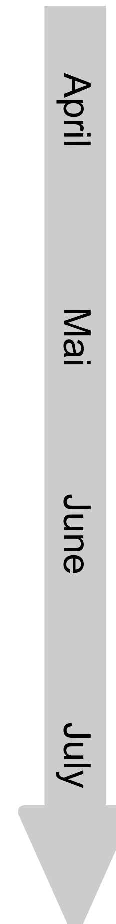
- … no weak force as we know it.
- … no Standard Model.
- … no **Lecture on Higgs Physics**.

Vorlesung Teilchenphysik II – Higgsphysik

- Vorlesung: 2 SWS, Übungen 1 SWS.
- Wahlfach im Masterstudium Physik, als Teilmodul eines Vertiefungs- bzw Ergänzungsfaches (6 LP) mit mündlicher Modulprüfung
- **Lehrveranstalltung:** 4022181.
- **Einordnung in Studiengang:** Master Physik, Bereich Teilchenphysik.
- **Leistungspunkte:** 6.
- **Semesterwochenstunden:** 2+1=3.
- **Literatur:** siehe Modulhandbuch. Weitere interessante Literatur wird in den jeweiligen Vorlesungen bekannt gegeben.
- Details entnehmen Sie bitte aus dem vorliegenden Modulhandbuch

Lecture Program

- Recall of prerequisites: Dirac-Eq, Klein-Gordon Eq, local gauge invariance (1 lecture, today)
- Review of what all this is about: SM of particle physics (1 lecture).
- Spontaneous symmetry breaking, Higgs meachanism (1 lecture).
- Lagrangian → observale (2 lectures).
- Accelerator/experiment → measurement (2 lectures).
- What we knew before the advent of the LHC (1 lecture).
- Higgs discovery & properties known by today (3 lectures).
- Higgs future and spinning around... (1 lectures).



Nota Bene

Theoretical Particle Physics II

Experimental Particle Physics I

Statistical Data Analysis

- Nobody left behind.
- Don't be boring at the same time.

- Try to be complete but specific.
- Try to give an interesting clue with each topic that we address.

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Theoretical Particle Physics II

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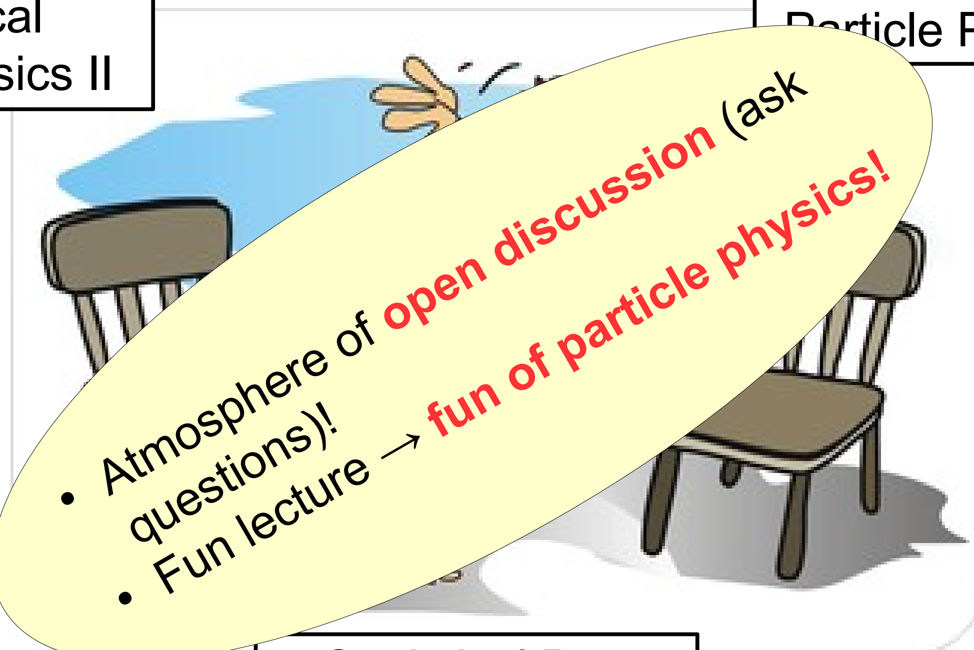

- Atmosphere of **open discussion** (ask questions)!
- Fun lecture → **fun of particle physics!**

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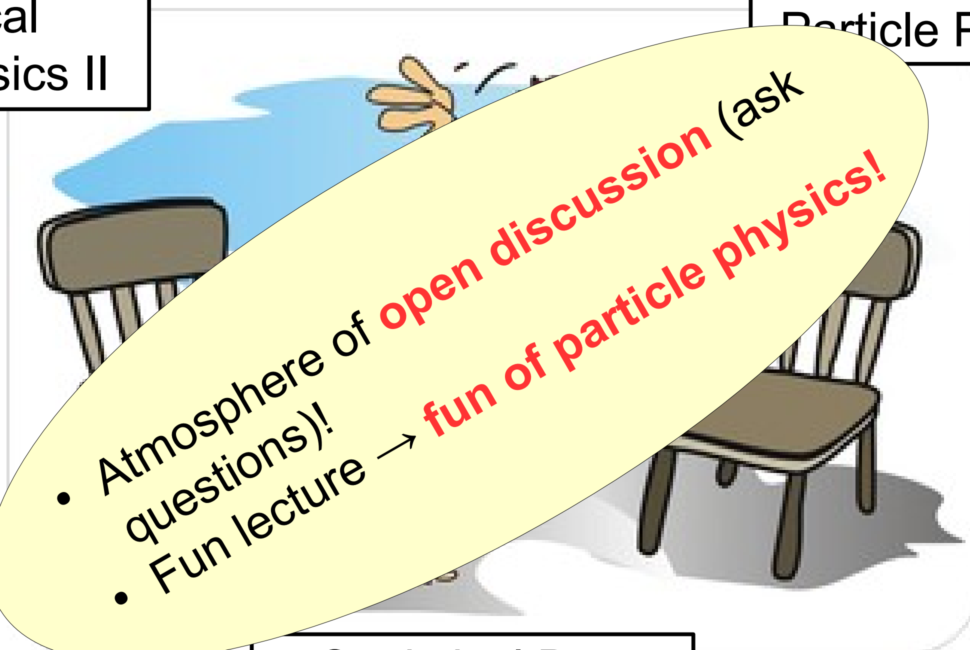

- Atmosphere of **open discussion** (ask questions)!
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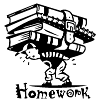
Statistical Data Analysis

This is an interactive lecture:



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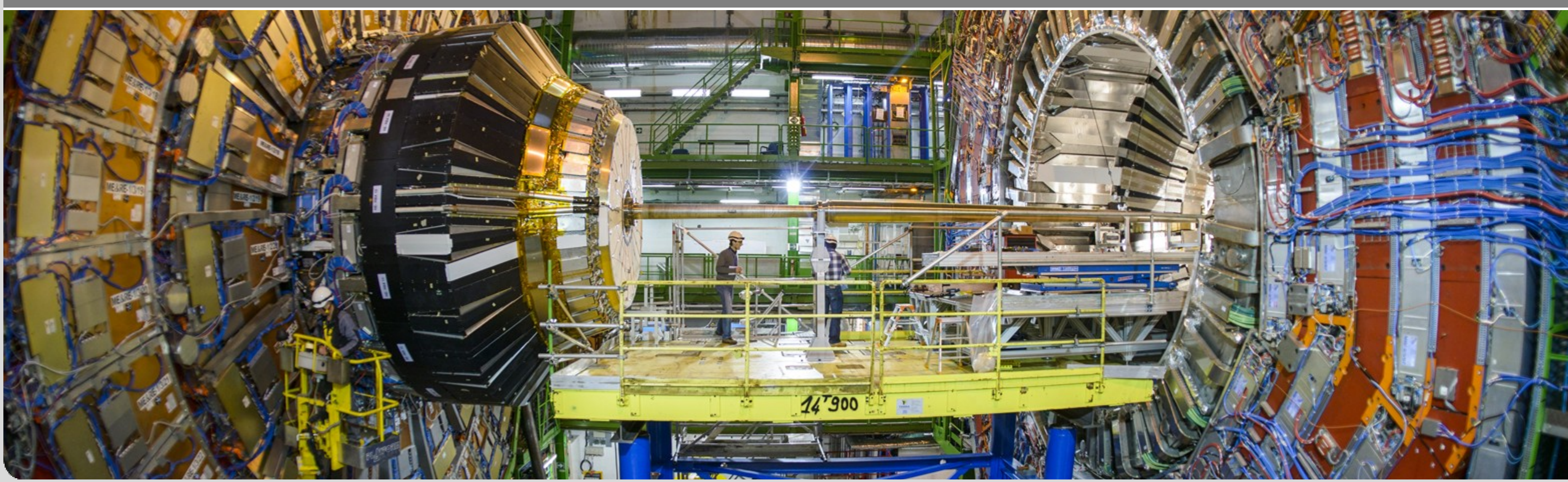
Relativistic Quantum Mechanics, Lagrange Formalism & Gauge Theories

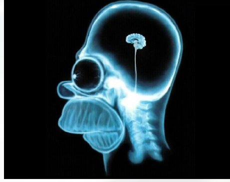
Roger Wolf

17. April 2014

INSTITUTE OF EXPERIMENTAL PARTICLE PHYSICS (IEKP) – PHYSICS FACULTY



Quiz of the Day



- What is the difference between a scalar, a Lorentz vector and a spinor?
- Deeper understanding of what local gauge invariance means.
- How do I know that a gauge boson is a boson?

Schedule for Today

1

Review of Relativistic QM:

- Klein-Gordon Eq
- Dirac Eq

2

Bosons & Fermions

3

Lagrange Formalism & Gauge Transformations:

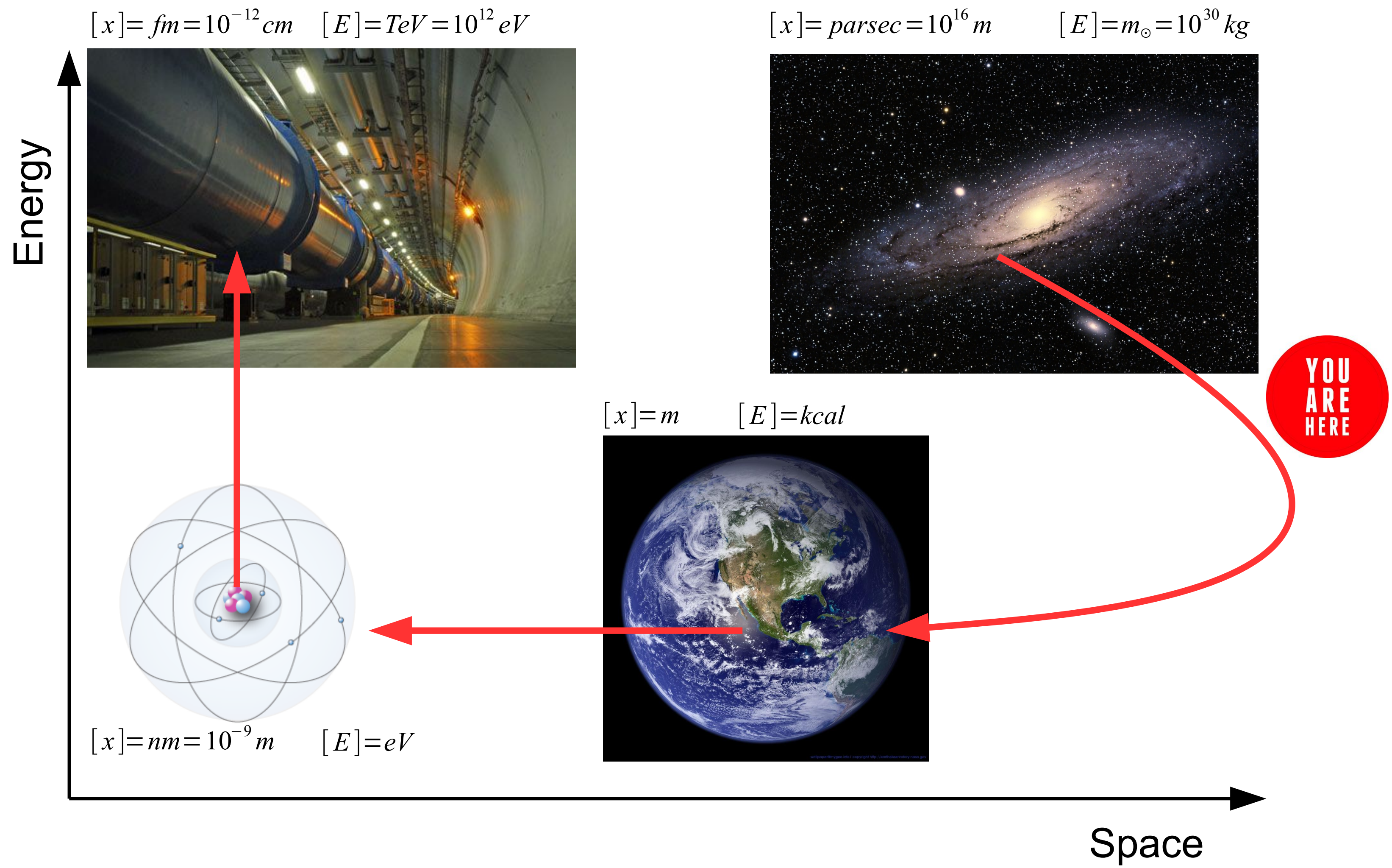
- Global / Local Gauge Transformations
- (Free) Gauge Fields

Review of Relativistic QM

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

A. Einstein

Scales: Between Cosmos & Particle Physics



Relativistic Quantum Mechanics

$[x] = fm = 10^{-12} cm \quad [E] = TeV = 10^{12} eV$

Smallest scales (10^{-12} cm).	+	Largest energies (10^{+12} eV).

(→ *Quantum Mechanics*)

(→ *Relativistic Dispersion Relation* $E^2 = p^2 + m^2$)

- Most important Eq's to describe particle dynamics: *Klein-Gordon*, *Dirac* Eq.

Relativistic Quantum Mechanics

Karlsruhe Institute of Technology

$[x] = fm = 10^{-12} cm \quad [E] = TeV = 10^{12} eV$

Natural units ($\rightarrow \hbar = 1, c = 1$):

$$[m] = \mathrm{GeV} \qquad [x] = \mathrm{GeV}^{-1}$$
$$[E] = \mathrm{GeV} \qquad [t] = \mathrm{GeV}^{-1}$$
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$$\Delta p \cdot \Delta x \gtrsim \hbar$$

($\rightarrow$ *Uncertainty Relation*)

Smallest scales (10^{-12} cm).

($\rightarrow$ *Quantum Mechanics*)

+

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($\rightarrow$ *Relativistic Dispersion Relation* $E^2 = p^2 + m^2$)

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Klein-Gordon Equation

- Motivation:

Canonical Operator Replacement

Application to Wave Function

$$E^2 = p^2 + m^2 \quad \Rightarrow \quad \boxed{\begin{array}{l} E \to i\partial_t \\ \vec{p} \to -i\vec{\nabla} \end{array}} \quad \Rightarrow \quad \boxed{\left(\partial_\mu \partial^\mu + m^2\right) \phi = 0}$$

(Klein-Gordon Eq)

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(Klein-Gordon Eq)

- Solutions:

$$\phi_+(\vec{x},t) = u(\vec{p})e^{+i(\vec{p}\vec{x}-Et)}$$

$$\phi_-(\vec{x},t) = v(\vec{p})e^{-i(\vec{p}\vec{x}-Et)} \qquad E(\vec{p}) = \sqrt{\vec{p}^2 + m^2}$$

(Free Wave)

Klein-Gordon Equation

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(Free Wave)

- Peculiarity:

$$\hat{H}_0 = \sqrt{m^2 - \vec{\nabla}^2}$$

(Non-Local)

non-local operator.

Klein-Gordon Equation

Karlsruhe Institute of Technology

- Motivation:

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- Peculiarity:

$$\hat{H}_0 = \sqrt{m^2 - \vec{\nabla}^2} = m\sqrt{1 - \frac{\vec{\nabla}^2}{m^2}} = m - \frac{\vec{\nabla}^2}{2m} + \cdots$$

(Non-Local)

→ non-local operator.

Dirac Equation: Motivation

- Historical approach by *Paul Dirac 1927*:

  Find representation of relativistic dispersion relation, which is linear in space time derivatives:

$$i\partial_t\psi = \hat{H}_0\psi = \left(-i\vec{\alpha}\vec{\nabla} + \beta m\right)\psi$$

- Cannot be pure numbers. *Algebraic operators*.
- Need four independent operators.

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Require Klein-Gordon Eq to be fulfilled for a free Dirac particle:

$$(i\partial_t)^2\psi = \left(-\vec{\alpha}\vec{\nabla} + \beta m\right)^2\psi$$

$$= \Big[\underbrace{(\alpha_i\alpha_j + \alpha_j\alpha_i)}_{\{\alpha_i,\alpha_j\} = 2\delta_{ij}}\,\partial_i\partial_j{}_{(i\leq j)} - im\,\underbrace{(\alpha_i\beta + \beta\alpha_i)}_{\{\alpha_i,\beta\} = 0}\,\partial_i + \underbrace{(\beta m)^2}_{\beta^2 = 1}\Big]\psi \stackrel{!}{\equiv} \Big[-\vec{\nabla}^2 + m\Big]\psi$$

Anti-Commutator Relations.

Dirac Equation: General Properties of $\vec{\alpha}$ and β

Karlsruhe Institute of Technology

- Operators $\vec{\alpha}$ and β can be expressed by matrices:

Must be hermitian since $\hat{H}_0$ should have real *eigenvalues*.

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Karlsruhe Institute of Technology

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Must be hermitian since $\hat{H}_0$ should have real *eigenvalues*.

Must be traceless:

$$Tr(\alpha_i) = Tr(\alpha_i \underbrace{\beta\beta}_{\mathbb{I}}) \underset{\text{cyclic permutation}}{=} Tr(\beta\alpha_i\beta) \underset{\text{anti-commutator relation}}{=} -Tr(\beta\beta\alpha_i) = -Tr(\alpha_i) = 0$$

cyclic permutation *anti-commutator relation*

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Must have at least dim=4:

- $\alpha_i^2 = \mathbb{I} \rightarrow$ has only eigenvectors ±1.
- $\beta^2 = \mathbb{I} \rightarrow$ has only eigenvectors ±1.
- Dimension must be even to obtain 0 trace.
- $\mathbb{I}$ + Pauli matrices $(\mathbb{I}, \sigma_i)$ form a basis of the space of 2×2 matrices. But $\mathbb{I}$ is not traceless.
- Simplest representation must at least have dim=4 (can be higher dimensional though).

Dirac Equation: Concrete Representations

- α_i and β matrices (in *Dirac* representation):

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

($\sigma_i (i = 1, 2, 3)$ *are the Pauli Matrices*)

- γ^μ matrices:

$$\gamma^0 \equiv \beta \qquad \gamma^i \equiv \beta\alpha_i \quad \Rightarrow \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

(*Compact Notation of Algebra*)

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

Institute of Experimental Particle Physics (IEKP)

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$\mathbb{I}_4$	1 matrix
γ^μ	4 matrices
$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$	6 matrices
$\gamma^\mu\gamma^5$	4 matrices
$\gamma^5 \equiv \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$	1 matrix

- Basis of 4×4 matrices.
- Orthonormal (with product $< .|. >= \frac{1}{4}Tr(\cdot\,\cdot)$).
- Traceless (apart from $\mathbb{I}_4$).

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For spacial components of $\sigma^{\mu\nu}$:

$$\Sigma^k = \frac{1}{2}\epsilon^{klm}\sigma^{lm} \qquad (k = 1, 2, 3)$$

$$\left.\begin{aligned} &\left[\frac{1}{2}\Sigma^k, \frac{1}{2}\Sigma^l\right] = i\epsilon^{klm}\left(\frac{1}{2}\Sigma^m\right) \\ &\sum_{k=1}^{3}\left(\frac{1}{2}\Sigma^m\right)^2 = \frac{1}{2}\left(\frac{1}{2}+1\right) \end{aligned}\right\} \text{Spin Algebra}$$

$$\Sigma^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

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Dirac Equation: Solutions

- Final formulation: $\boxed{(i\gamma^\mu\partial_\mu - m)\,\psi = 0}$ *(Dirac Eq)*
- Solutions:

$$\psi_+(\vec{x}) = u(\vec{p})e^{+i(\vec{p}\vec{x}-Et)}$$

$$\psi_-(\vec{x}) = v(\vec{p})e^{-i(\vec{p}\vec{x}-Et)} \qquad E(\vec{p}) = \sqrt{\vec{p}^2 + m^2}$$ *(Free Wave)*

at rest: $\vec{p} \equiv 0$

$$u_\uparrow(0) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad u_\downarrow(0) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad \Bigg\} \ +m \text{ solution}$$

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$$\Lambda : (m,0,0,0) \to (E, p_x, p_y, p_z)$$ *(Lorentz Transformation)*

at rest: $\vec{p} \equiv 0$ in motion: $\vec{p} \neq 0$

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$$u_\uparrow(\vec{p}) = \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix}E+m\\0\\p_z\\p_x+ip_y\end{pmatrix} \quad u_\downarrow(\vec{p}) = \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix}0\\E+m\\p_x-ip_y\\-p_z\end{pmatrix}$$

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(Dirac-)Spinors

- $\psi(\vec{x}, t)$ is a *Spinor*:

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Rotation with $\vec{\varphi}$.

Rotation by 2π lead to $-\psi(\vec{x},t)$.

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Boost with velocity $\vec{v_b}$.

$$S(\Lambda) = e^{\vec{v_b}\cdot\frac{1}{2}\vec{\alpha}} = \cosh\left(\frac{v_b}{2}\right) + \sinh\left(\frac{v_b}{2}\right)\left(\hat{v_b}\cdot\vec{\alpha}\right)$$

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Karlsruhe Institute of Technology

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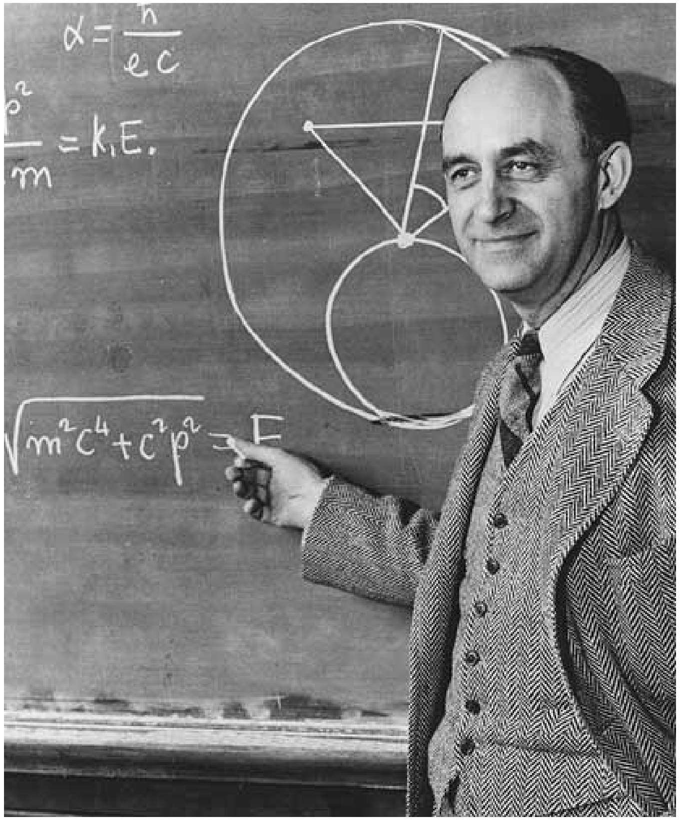
$\overline{\psi}\psi$	Scalar
$\overline{\psi}\gamma^5\psi$	Pseudo Scalar
$\overline{\psi}\gamma^\mu\psi$	Vector
$\overline{\psi}\gamma^5\gamma^\mu\psi$	Axial Vector
$\overline{\psi}\sigma^{\mu\nu}\psi$	Tensor (2. order)

$\overline{\psi} = \psi^\dagger\gamma^0$ *(Adjoint Spinor)*

Bosons & Fermions

Satyenda Nath Bose

(*1. January 1894, † 4. February 1974)



Enrico Fermi

(*29. September 1901, † 28. November 1954)

Bosons	Fermions
$$\left(\partial_\mu \partial^\mu + m^2\right) \phi = 0$$	$$\left(i\gamma^\mu \partial_\mu - m\right) \psi = 0$$
• Integer spin 0, 1, ...[(1)]	• Half-integer spin ½, ...[(1)]
• Commutator relations [. , .].	• Anti-commutator relations { . , . }.

[(1)] This holds for elementary particle as well as for pseudo-particles.

Bosons

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Fermions

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- Half-integer spin ½, ...[1]
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Multi-particle systems

Bosons:

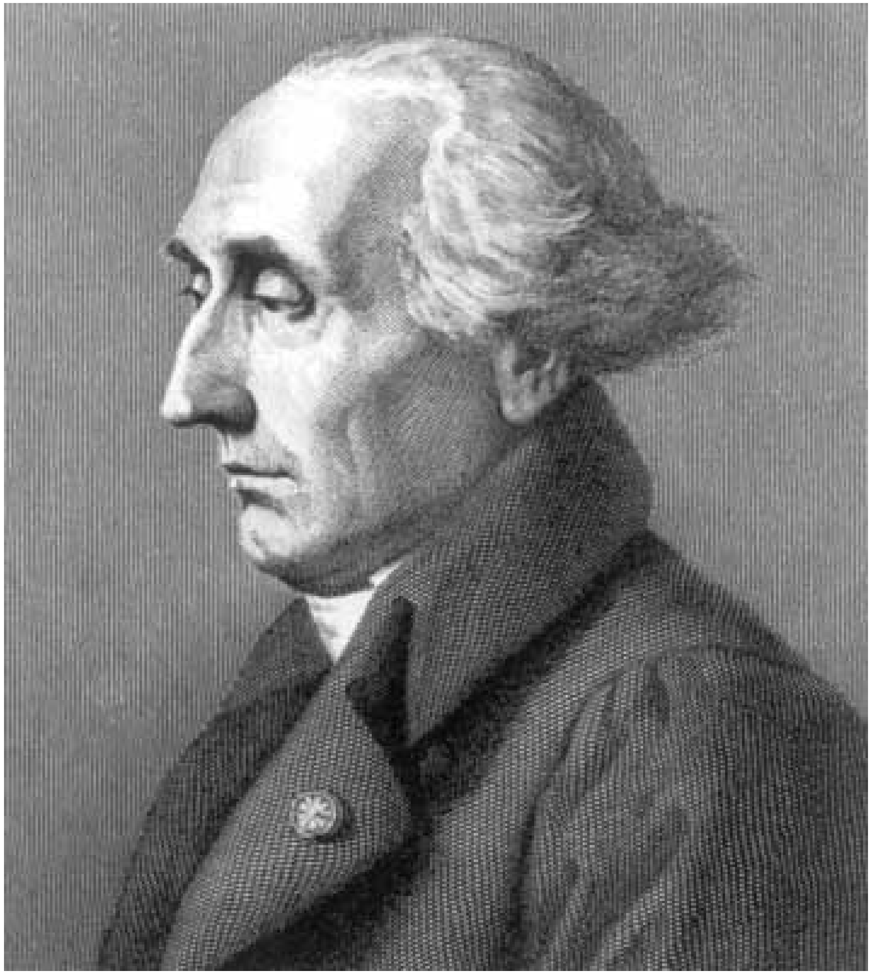
- Symmetric wave functions.
- Bose-Einsten statistics.
- More than one particle can be described by single wave function (*e.g. …?!?*).

Fermions:

- Anti-symmetric wave functions.
- Fermi statistics.
- Each particle occupies unique place in phasespace (*Pauli Principle*).

[1] This holds for elementary particle as well as for pseudo-particles.

Lagrange Formalism & Gauge Transformations



Joseph-Louis Lagrange
(*25. January 1736, † 10. April 1813)

Lagrange Formalism (Classical Field Theories)

- All information on a physical system is contained in the *Action* integral:

$$S = \int \mathcal{L}(\partial_\mu \phi, \phi) \mathrm{d}x_\mu$$

Action: S

Field: ϕ *(Generalization of Canonical Coordinates)*

Lagrange Density: $\mathcal{L}$

($\sum E_{\mathrm{kin}} - \sum E_{\mathrm{pot}}$ *)*

- Equations of motion derived from the *Euler-Lagrange Formalism*:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

(From Variation of Action)

- **NB:** What is the dimension of $\mathcal{L}$?

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- **NB:** What is the dimension of $\mathcal{L}$? $\longrightarrow$ $\mathcal{L}$ has the dimension GeV^4 .

Lagrange Density for Free Bosons & Fermions

For Bosons:

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^*$$

For Fermions:

$$\mathcal{L} = \overline{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi$$

- Proof by applying *Euler-Lagrange Formalism* (shown only for Bosons here):

$$\partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^*)} - \frac{\partial \mathcal{L}}{\partial \phi^*} = 0$$

$$\partial^\mu \partial_\mu \phi \qquad -m^2 \phi \longrightarrow \left(\partial^\mu \partial_\mu + m^2 \right) \phi = 0$$

- **NB:**
  - There is a distinction between $\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$ and $\partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^*)}$.
  - Most trivial is variation by $\overline{\psi}$, least trivial is variation by ψ.

Global Phase Transformations

- The Lagrange density is covariant under global phase transformations (shown here for the fermion case only):

$$\psi(\vec{x},t) \rightarrow \psi'(\vec{x},t) = e^{i\vartheta}\psi(\vec{x},t)$$

$$\overline{\psi}(\vec{x},t) \rightarrow \overline{\psi}'(\vec{x},t) = \overline{\psi}(\vec{x},t)e^{-i\vartheta}$$

(Global Phase Transformation)

$$\vartheta = const$$

$$\mathcal{L}' = \overline{\psi}'\left(i\gamma^\mu\partial_\mu - m\right)\psi' = \overline{\psi}e^{-i\vartheta}\left(i\gamma^\mu\partial_\mu - m\right)e^{i\vartheta}\psi$$

$$= \overline{\psi}\left(i\gamma^\mu\ \partial_\mu \qquad - m\right)\psi \quad = \mathcal{L}$$

- Here the phase ϑ is fixed at each point in space $\vec{x}$ at any time t .
- What happens if we allow different phases at each point in $(\vec{x},t)$?

Local Phase Transformations

- The Lagrange density is ~~covariant~~ under local phase transformations (shown here for the fermion case only):

$$\boxed{\begin{aligned}\psi(\vec{x},t) &\to \psi'(\vec{x},t) = e^{i\vartheta}\psi(\vec{x},t)\\ \overline{\psi}(\vec{x},t) &\to \overline{\psi}'(\vec{x},t) = \overline{\psi}(\vec{x},t)e^{-i\vartheta}\end{aligned}}$$

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Connects neighboring points in $(\vec{x},t)$

Breaks invariance due to ∂_μ in $\mathcal{L}$. $\longrightarrow \frac{\psi(x+\Delta x)-\psi(x)}{\Delta x}$

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- Local covariance can be enforced by introduction of a covariant derivative $\partial_\mu \rightarrow D_\mu = \partial_\mu - ieA_\mu$ with an according transformation rule:

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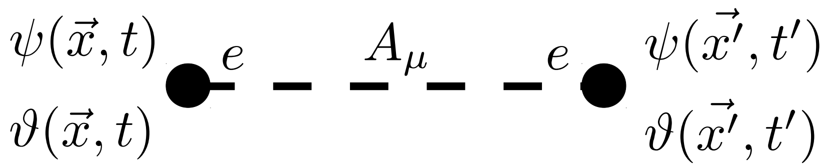
- **NB:** What is the transformation behavior of the gauge field A_μ?

$$A_\mu \rightarrow A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\vartheta \longrightarrow$$ known from electro-dynamics!

Gauge Field

- Possible to allow arbitrary phase ϑ of $\psi(\vec{x},t)$ at each individual point in $(\vec{x},t)$
- Requires introduction of a mediating field A_μ, which transports this information from point to point.



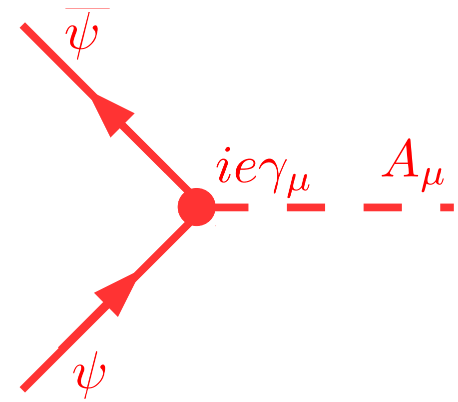
- The gauge field A_μ couples to a quantity e of the spinor field $\psi(\vec{x},t)$, which can be identified as the electric charge of the fermion.
- The gauge field A_μ can be identified with the photon field.

Interacting Fermion

- Introduction of covariant derivative leads to *Lagrange density* of interacting fermion with electric charge e:

$$\mathcal{L}_{\mathrm{IA}} = \overline{\psi}\,(i\gamma^\mu(D_\mu - m)\,\psi$$

$$= \underbrace{\overline{\psi}\,(i\gamma^\mu\ \partial_\mu - m)\,\psi}_{\text{Free Fermion Field}} - \underbrace{ie\overline{\psi}\gamma^\mu A_\mu\psi}_{\text{IA Term}}$$



- For completion the dynamics for a free *gauge boson field* (=photon) are missing.

Free Gauge Field

- **Ansatz:**

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad\qquad \mathcal{L}_{\text{Kin}} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

(Field-Strength Tensor) *(Free Photon Field)*

- **Motivation:**

  - Variation of the action integral

    $\mathcal{S} = \delta \int (-m\mathrm{d}s - eA_\mu \mathrm{d}x^\mu)$

    in classical field theory, leads to

    $m\frac{dv^\mu}{ds} = e(\partial_\mu A_\nu - \partial_\nu A_\mu)v^\nu$

  - Can also be obtained from:

    $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) = \frac{i}{e}[D_\mu, D_\nu]$

  - $F_{\mu\nu}F^{\mu\nu}$ is Lorentz invariant.
  - A_μ appears quadratically → linear appearance in variation that leads to equations of motion (→ superposition of fields).
  - $F_{\mu\nu}$ is gauge invariant.

Complete Lagrange Density

- Application of $U(1)$ gauge symmetry leads to Largange density of QED:

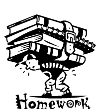
$$
\begin{aligned}
\mathcal{L}_{\text{QED}} &= \overline{\psi}\,(i\gamma^\mu(D_\mu - m)\,\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
&= \underbrace{\overline{\psi}\,(i\gamma^\mu \;\; \partial_\mu - m)\,\psi}_{\text{Free Fermion Field}} - \underbrace{ie\overline{\psi}\gamma^\mu A_\mu \psi}_{\text{IA Term}} - \underbrace{\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{Gauge}}
\end{aligned}
$$

(Interacting Fermion)

- Variation of $\overline{\psi}$:

$$i\gamma^\mu\left(\partial_\mu - m\right)\psi - ie\gamma^\mu A_\mu \psi = 0$$

- Derive equations of motion for an interacting boson.



Complete Lagrange Density

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- Application of $U(1)$ gauge symmetry leads to Largange density of QED:

$$
\begin{aligned}
\mathcal{L}_{\text{QED}} &= \bar{\psi}\,(i\gamma^\mu(D_\mu - m)\,\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
&= \underbrace{\bar{\psi}\,(i\gamma^\mu\ \ \partial_\mu - m)\,\psi}_{\text{Free Fermion Field}} - \underbrace{ie\bar{\psi}\gamma^\mu A_\mu\psi}_{\text{IA Term}} - \underbrace{\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{Gauge}}
\end{aligned}
$$

(Interacting Fermion)

- Variation of A_μ :

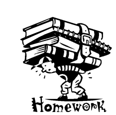
$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = \partial_\mu F^{\mu\nu} = 0$$

$$\partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = (\partial_\mu\partial^\mu A_\mu - \partial^\nu \underbrace{\partial_\mu A^\mu}_{\partial_\mu A^\mu = 0}) = 0$$

(Lorentz Gauge)

$$\boxed{\left(\partial_\mu\partial^\mu - 0\right)A_\mu = 0}$$

(Klein-Gordon Equation for a Massless Particle)



Concluding Remarks

- Principle of local gauge invariance leads to structure for particle interaction that corresponds to QED.
- Gauge invariance is a geometrical phenomenon.
- Explicitly shown that the gauge field is a boson with zero mass.

Sneak Preview for Next Week

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- Simple phase transformations $e^{i\vartheta}$ correspond to the $U(1)$ symmetry group.
- Discuss how local gauge invariance requirements corresponding to more complex symmetry groups will lead to the wealth of possible interactions in the SM.
- Short sketch of the SM (emphasize electroweak sector, still w/o masses).

Further Reading

- Bjorken/Drell “*Relativistic Quantum Mechanics*”.
- Aichinson/Hey: “*Gauge Theories and Particle Physics (Volume 1)*”.
- Lifschitz/Landau: “*Classical Field Theory (Volume 2 of lectures)*”.