Exercises to Lecture 1: Reprise of Relativistic Quantum Mechanics, Lagrange Formalism & Gauge Theories

Exercise 1 (Dirac equation from Lagrangian density):

In the lecture we have derived the *Klein-Gordon* equation from the corresponding Lagrangian density

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^*$$

using the *Euler-Lagrange* equations applied to the field ϕ^*. Do the same exercise to obtain the *Dirac* equation by applying the *Euler-Lagrange* equations on the field $\overline{\psi}$.

Exercise 2 (Gauge invariance of $F_{\mu\nu}$):

In the lecture we have made the *ansatz*

$$\mathcal{L}_{\mathrm{kin}} = \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

for the Lagrangian density term that corresponds to the kinetic term of the gauge field in the full Lagrangian density. Using the translation behavior of the gauge field

$$A_\mu \to A'_\mu = A_\mu + \tfrac{1}{e} \partial_\mu \vartheta$$

proof that $F'_{\mu\nu} = F_{\mu\nu}$. As a consequence the term $\mathcal{L}_{\mathrm{kin}}$ is Lorentz invariant and gauge invariant (i.e. it does have the required transformation behavior.)

Exercise 3 (Local Gauge Invariance for Bosons):

In the lecture we have sketched the exercise to enforce local gauge invariance for the example of fermions, starting from the Lagrangian density term for fermions. We have seen how the requirement of local gauge invariance leads to the full Lagrangian of Quantum Electrodynamics (QED). In nature there exist also charged bosons, so the same procedure should work for bosons as well. Proof that the same covariant derivative with the same gauge transformation laws works equally well for bosons as for fermions:

a)
Translate the transformation behavior for fermions to bosons

$$\begin{aligned}
\phi(\vec{x},t) &\to \phi'(\vec{x},t) &= e^{i\vartheta} \phi(\vec{x},t) \\
\phi^*(\vec{x},t) &\to \phi^{*\prime}(\vec{x},t) &= \phi^*(\vec{x},t) e^{-i\vartheta} \\
D_\mu &\to D'_\mu &= D_\mu - i \partial_\mu \vartheta
\end{aligned}$$

apply it to the Lagrangian density term for bosons and proof the relation:

$$\mathcal{L}' = D'_\mu \phi' (D'^\mu \phi')^* - \tfrac{m}{2} \phi' \phi'^* = D_\mu \phi (D^\mu \phi)^* - \tfrac{m}{2} \phi \phi^* = \mathcal{L}$$

Note that for this exercise you have to take care of the complex conjugation of D^μ, which is trivially not the case for ∂^μ.

b)
Write out the full Lagrangian density term for bosons in analogy to the Lagrangian density term $\mathcal{L}_{\text{QED}}$ that has been given in the lecture. (You can add the term for the free gauge field for completeness, if you like, but this is not important for the point that we want to make here.) Derive the equations of motion starting from this Lagrangian density term and compare it with the fermion case that you have seen in the lecture.

Exercise 4 (Variation of the Free Gauge Field A_μ):

In the lecture we have shown how from the variation of the free gauge field term

$$\mathcal{L}_{\text{kin}} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

the *Klein-Gordon* equation for a free massless boson follows, which can be shown in the physical *Lorentz gauge* of electrodynamics. Try to follow the line of arguments step by step starting from the *Euler-Lagrange* equations:

$$\partial_\mu \frac{\partial \mathcal{L}_{\text{Kin}}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}_{\text{Kin}}}{\partial \phi} = 0$$

Especially proof the (non-trivial) missing piece that we have not shown in the lecture:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \partial_\mu F^{\mu\nu}$$