From Lagrange Density to Observable

Roger Wolf

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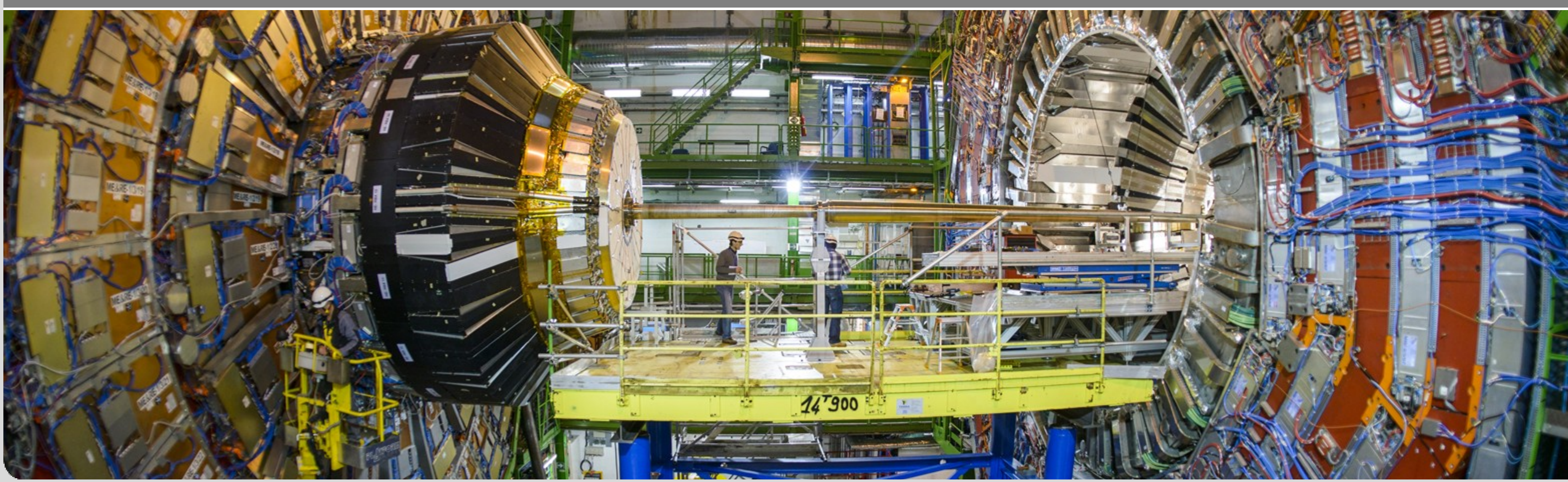
INSTITUTE OF EXPERIMENTAL PARTICLE PHYSICS (IEKP) – PHYSICS FACULTY



Schedule for Today

3

Boundaries on the Higgs boson mass within the SM

2

From Lagrangian to observable (on trees and loops).

1

Milestones in the formulation of the SM & discussion

Quiz of the Day

- What is a propagator?
- What is the connection between the scattering matrix element $\mathcal{S}_{fi}$ and the Lagrangian density, we were discussing during the last lectures?
- Does a *Feynman* graph have a time direction? If yes, what is it?

SM (all inclusive): Wrap it up!

Step 1: Electroweak Interactions

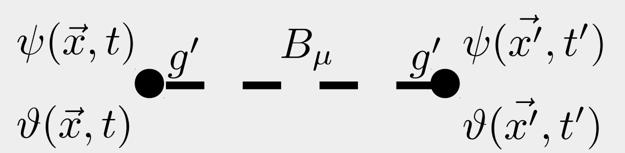
- Combine ν and e_L into a $SU(2)$ doublet, which behaves like a vector in *weak isospin* space. Enforce local gauge invariance for $\mathcal{L}$. The e_R component of the electron behaves like a $SU(2)$ singlet.

$$\psi_L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L \quad e_R$$

$$D_\mu = (\partial_\mu + igW_\mu)$$

- Description of weak interactions.
- Gauge bosons W^a_μ.

- To also obtain a description of the electromagnetic force additionally local gauge invariance is enforced for the $U(1)$ symmetry on the doublet as a whole and on the singlet.



- Description of electromagnetic interactions (W^a_μ & B_μ).

Step 2: Weinberg Rotation

- To achieve that the coupling to the ν is governed only by a single physical field, the fields W_μ^3 and B_μ are rotated by the Weinberg angle θ_W.

- Obtain physical fields (Z_μ & A_μ).

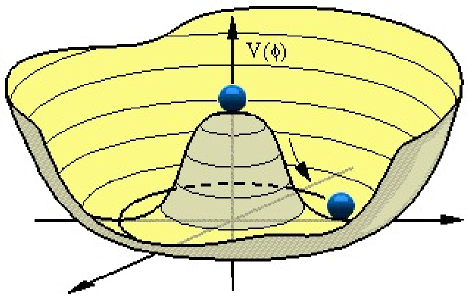
$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix}$$

$$\sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}} \qquad \cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}$$

Step 3: Higgs Mechanism

- To obtain mass terms for the massive gauge bosons introduce a new field ϕ with a potential that leads to spontaneous symmetry breaking for this field. The gauge fields are coupled to ϕ via the covariant derivative $D_\mu \phi$.

- Masses for gauge bosons (m_Z & m_W).
- Massive Higgs boson H.
- Couplings of gauge bosons to H $\propto m_{W/Z}^2 H$.



- To obtain mass terms for fermions couple the fermion fields to ϕ via *Yukawa* couplings.

- Couplings of fermions $\propto m_f H$.

SM Full Lagrangian

$$\mathcal{L}^{\text{SM}} = \mathcal{L}^{\text{Lepton}}_{\text{kin}} + \mathcal{L}^{CC}_{\text{IA}} + \mathcal{L}^{NC}_{\text{IA}} + \mathcal{L}^{\text{Gauge}}_{\text{kin}} + \mathcal{L}^{\text{Higgs}}_{kin} + \mathcal{L}^{\text{Higgs}}_{V(\phi)} + \mathcal{L}^{\text{Higgs}}_{\text{Yukawa}}$$

$$\mathcal{L}^{\text{Lepton}}_{\text{kin}} = i\bar{e}\gamma^{\mu}\partial_{\mu}e + i\bar{\nu}\gamma^{\mu}\partial_{\mu}\nu$$

$$\mathcal{L}^{CC}_{\text{IA}} = -\frac{e}{\sqrt{2}\sin\theta_W}\left[W^{+}_{\mu}\bar{\nu}\gamma_{\mu}e_L + W^{-}_{\mu}\bar{e}_L\gamma_{\mu}\nu\right]$$

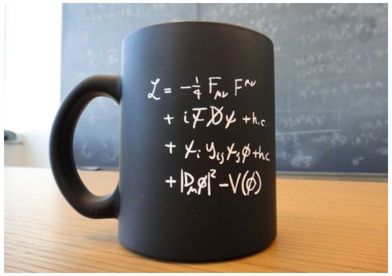
$$\mathcal{L}^{NC}_{\text{IA}} = -\frac{e}{2\sin\theta_W\cos\theta_W}Z_{\mu}\left[(\bar{\nu}\gamma_{\mu}\nu) + (\bar{e}_L\gamma_{\mu}e_L)\right] - e\left[A_{\mu} + \tan\theta_W Z_{\mu}\right](\bar{e}\gamma_{\mu}e)$$

$$\mathcal{L}^{\text{Gauge}}_{\text{kin}} = -\frac{1}{2}Tr\left(W^{a}_{\mu\nu}W^{a\mu\nu}\right) - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}\Bigg|\ \begin{matrix} B_{\mu} \to A_{\mu} \\ W^{3}_{\mu} \to Z_{\mu} \end{matrix}$$

$$\mathcal{L}^{\text{Higgs}}_{kin} = \frac{1}{2}\partial_{\mu}H\partial^{\mu}H + \left(1 + \frac{1}{v}\frac{H}{\sqrt{2}}\right)^{2} m_W^2 W^{+}_{\mu}W^{\mu -} + \left(1 + \frac{1}{v}\frac{H}{\sqrt{2}}\right)^{2} m_Z^2 Z_{\mu}Z^{\mu}$$

$$\mathcal{L}^{\text{Higgs}}_{V(\phi)} = -\frac{\mu^2 v^2}{2} + \mu^2\left(\frac{H}{\sqrt{2}}\right)^{2} + 2\frac{\mu^2}{v}\left(\frac{H}{\sqrt{2}}\right)^{3} + \frac{\mu^2}{2v^2}\left(\frac{H}{\sqrt{2}}\right)^{4}$$

$$\mathcal{L}^{\text{Higgs}}_{\text{Yukawa}} = -\left(1 + \frac{1}{v}\frac{H}{\sqrt{2}}\right)m_e^2\bar{e}e$$



Questions???

- Is there any further questions or need for discussion on your side that we can address in the scope of this lecture?

Recommendation: read through the last three lectures once again in peace. You can have the same calculations with detailed explanations from front to end from:

The Higgs Boson Discovery at the Large Hadron Collider - Chapter 2.

Lagrangian Density → Observable

$\mathcal{L}$

Lagrangian Density → Observable

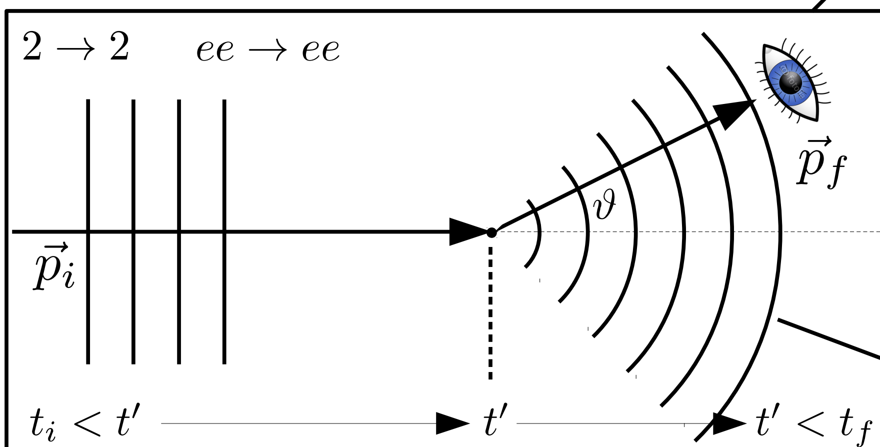
- Review the QM model of scattering wave.
- Turning the Dirac equation from a differential equation into an integral equation (→ Green's functions).
- Iterative solution of the integral equation with the help of perturbation theory.
- Finding the solution for A_μ when the target particle is moving (→ photon propagator).
- 1st order full solution and the Feynman rules.

QM Model of Particle Scattering

- Consider incoming collimated beam of projectile particles on target particle:

Scattering matrix $\mathcal{S}$ transforms initial state wave function ϕ_i into scattering wave ψ_{scat} ($\psi_{\text{scat}} = \mathcal{S} \cdot \phi_i$).

Observation (in $\Delta\Omega$): projection of plain wave ϕ_f out of spherical scattering wave ψ_{scat}.



Spherical scattering wave ψ_{scat}.

Initial particle: described by plain wave ϕ_i.

Localized potential.

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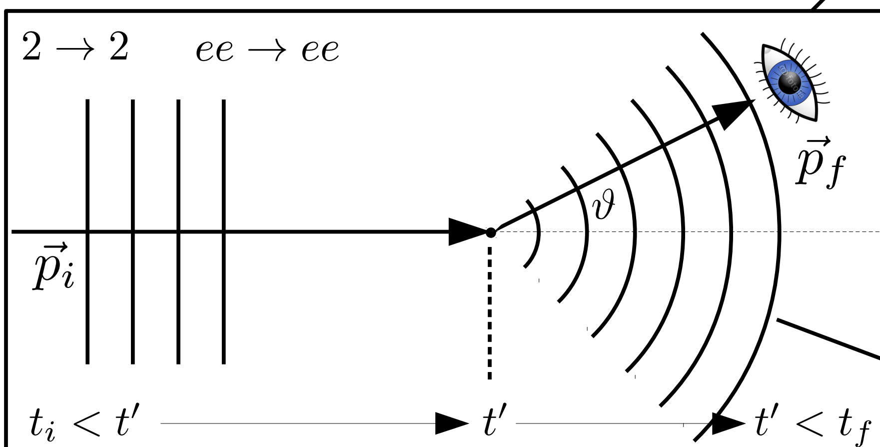

Observation probability:

$$\mathcal{S}_{fi} = \phi_f^\dagger \cdot \psi_{\text{scat}}$$
$$= \phi_f^\dagger \cdot \mathcal{S} \cdot \phi_i$$

Spherical scattering wave ψ_{scat} .

Initial particle: described by plain wave ϕ_i.

Localized potential.

Solution for ψ_{scat}

- In the case of fermion scattering the scattering wave ψ_{scat} is obtained as a solution of the *Dirac* equation for an interacting field:

$$\boxed{(i\gamma^\mu \partial_\mu - m)\,\psi_{\text{scat}} = -e\gamma^\mu A_\mu \psi_{\text{scat}}} \quad (+)$$

- The inhomogeneous *Dirac* equation is analytically not solvable.

Solution for ψ_{scat} (*Green's* Function)

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- The inhomogeneous *Dirac* equation is analytically not solvable. A formal solution can be obtained by the *Green's* Function $K(x-x')$:

$$\boxed{(i\gamma^\mu \partial_\mu - m)\,K(x-x') = \delta^4(x-x')}$$

$$\psi_{\text{scat}}(x) = -e \int K(x-x')\gamma^\mu A_\mu(x')\psi_{\text{scat}}(x')\mathrm{d}^4x'$$

$$(i\gamma^\mu \partial_\mu - m)\,\psi_{\text{scat}}(x) = -e \int \underbrace{(i\gamma^\mu \partial_\mu - m)\,K(x-x')}_{\delta^4(x-x')}\gamma^\mu A_\mu(x')\psi_{\text{scat}}(x')\mathrm{d}^4x'$$

$$= -eA_\mu(x)\psi_{\text{scat}}(x)$$

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$$\boxed{\psi_{\text{scat}}(x) = -e\int K(x-x')\gamma^\mu A_\mu(x')\psi_{\text{scat}}(x')\mathrm{d}^4x'}$$

NB: this is not a solution to (+), since ψ_{scat} appears on the left- and on the right-hand side of the equation. But it turns the differential equation into an integral equation.

Finding the *Green's* Function

Karlsruhe Institute of Technology

- The best way to find the *Green's* function is to use the *Fourier* transform:

$$K(x-x') = (2\pi)^{-4} \int \tilde{K}(p) e^{-ip(x-x')} \mathrm{d}^4 p \qquad \text{(\textit{Fourier} transform)}$$

  - Applying the *Dirac* equation to the *Fourier* transform of $K(x-x')$ turns the derivative into a product operator:

$$\underbrace{(i\gamma^\mu \partial_\mu - m) K(x-x')}_{\delta^4(x-x')} = (2\pi)^{-4} \int \underbrace{(\gamma^\mu p_\mu - m)\, \tilde{K}(p)}_{\mathbb{I}_4} e^{-ip(x-x')} \mathrm{d}^4 p$$

$$\delta^4(x-x') \equiv (2\pi)^{-4} \int \mathbb{I}_4 \, e^{-ip(x-x')} \mathrm{d}^4 p$$

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$$\delta^4(x-x') \equiv (2\pi)^{-4}\int \mathbb{I}_4\, e^{-ip(x-x')}\mathrm{d}^4p$$

- From the uniqueness of the *Fourier* transformation the solution for $\tilde{K}(p)$ follows:

$$(\gamma^\mu p_\mu - m)\,\tilde{K}(p) = \mathbb{I}_4$$

$$(\gamma^\mu p_\mu + m)\cdot(\gamma^\mu p_\mu - m)\,\tilde{K}(p) = (\gamma^\mu p_\mu + m)\cdot\mathbb{I}_4$$

The Fermion Propagator

- The *Fourier* transform of the *Green's* function is called *Fermion* propagator:

$$(\gamma^\mu p_\mu + m) \cdot (\gamma^\mu p_\mu - m)\, \tilde{K}(p) = (\gamma^\mu p_\mu + m) \cdot \mathbb{I}_4$$

$$\boxed{\tilde{K}(p) = \frac{(\gamma^\mu p_\mu + m)}{p^2 - m^2}}$$ (Fermion propagator)

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- The Fermion propagator is a 4×4 matrix, which acts in the Spinor room.
- It is only defined for virtual electrons since $p^2 - m^2 = E^2 - \vec{p}^2 - m^2 \neq 0$.

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- The *Green's* function can be obtained from $\tilde{K}(p)$ by:

$$K(x - x') = (2\pi)^{-4} \int \mathrm{d}^3\vec{p}\, e^{i\vec{p}(\vec{x}-\vec{x}')} \int_{-\infty}^{+\infty} \mathrm{d}p_0 \frac{(\gamma^\mu p_\mu + m)}{(p_0 - E)(p_0 + E)}\, e^{-ip_0(t-t')}$$

$$E = \sqrt{\vec{p}^2 + m^2}$$

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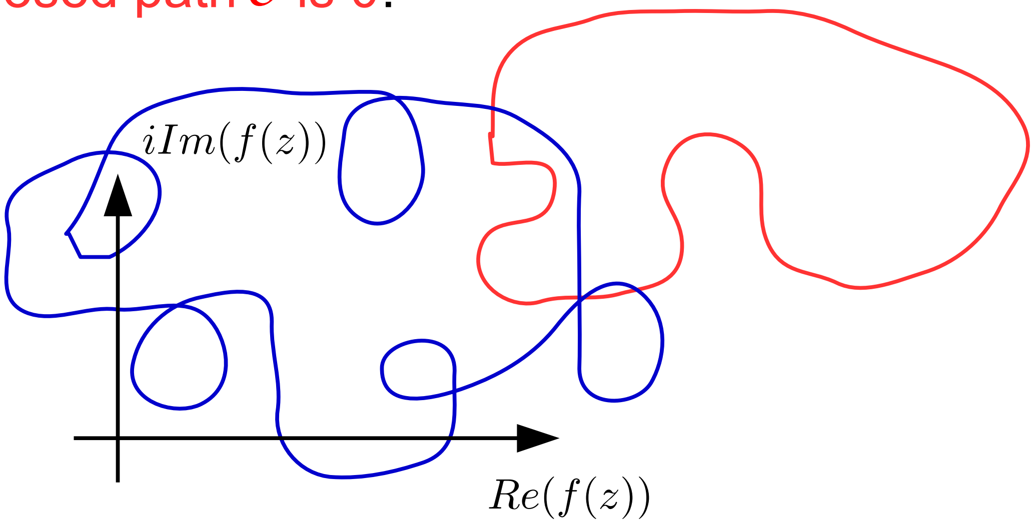
  - $K(x - x')$ has two poles in the integration plane (at $p_0 = \pm E$).
  - The integral can be solved with the methods of *function theory*.

Excursion into Function Theory

- When integrating a “well behaved” function w/o poles in the complex plain any path integral along a closed path $\mathcal{C}$ is 0:

Example: $f(z) = z^2$

$$\oint_{\mathcal{C}} z^2 \mathrm{d}z = 0$$



- When integrating a “well behaved” function with(!) poles in the complex plain the solution is $2i\pi\times$the “residual(s)” of the included poles:

Example: $f(z) = \frac{R}{z}$

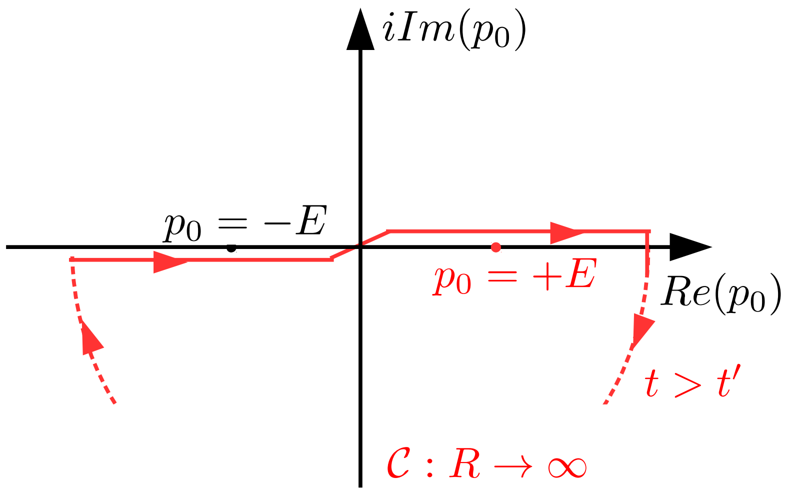
$$\oint_{\mathcal{C}} \frac{R}{z}\mathrm{d}z = 2i\pi \times R$$

Not matter how $\mathcal{C}$ is chosen, as long as it includes $z = (0 + i0)$.

cf. Freitag/Busam *Funktionentheorie*

The Fermion Propagator (Time Integration $t > t'$)

- Choose path $\mathcal{C}$ in complex plain to circumvent poles:



$$\int_{-\infty}^{+\infty} \mathrm{d}p_0 \, \frac{(\gamma^\mu p_\mu + m)}{(p_0 - E)(p_0 + E)} \, e^{-ip_0(t-t')}$$

- For $t > t'$ ($e^{-ip_0(t-t')} \to 0$ for $Im(p_0) \ll 0$):
  $\to$ close contour in lower plane & calculate integral from residual of enclosed pole.

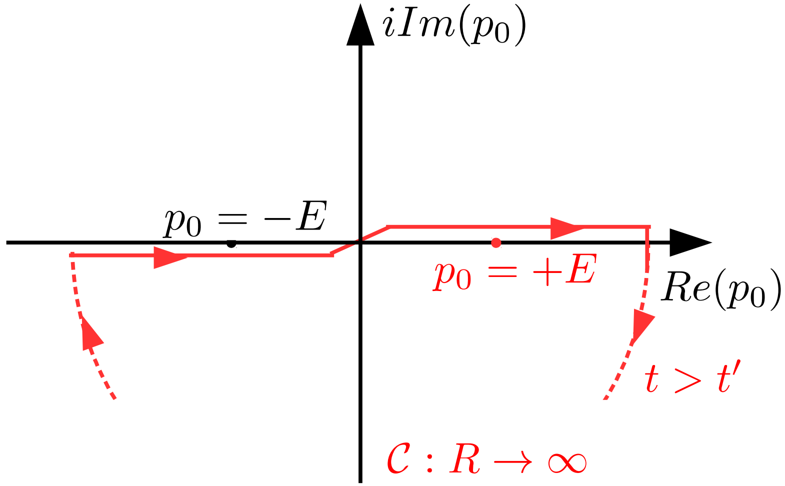
$$\oint_{\mathcal{C}} \mathrm{d}p_0 \underbrace{\frac{1}{p_0 - E}}_{\substack{\text{pole at:} \\ p_0 = +E}} \cdot \underbrace{\frac{(\gamma^\mu p_\mu + m)}{p_0 + E} \, e^{-ip_0(t-t')}}_{\text{residuum: } f(p_0)} = -2\pi i \cdot f(p_0)\big|_{p_0 = E}$$

Sign due to sense of integration.

The Fermion Propagator (Time Integration $t > t'$)

- Choose path $\mathcal{C}$ in complex plain to circumvent poles:

$$\boxed{\int_{-\infty}^{+\infty} \mathrm{d}p_0 \frac{(\gamma^\mu p_\mu + m)}{(p_0 - E)(p_0 + E)} \, e^{-ip_0(t-t')}}$$



$$\oint_{\mathcal{C}} \mathrm{d}p_0 \frac{1}{p_0 - E} \cdot \frac{(\gamma^\mu p_\mu + m)}{p_0 + E} \, e^{-ip_0(t-t')} = -2\pi i \cdot f(p_0)\big|_{p_0 = E}$$

$$\boxed{K(x - x') = -i(2\pi)^{-3} \int \mathrm{d}^3\vec{p} \, \frac{+\gamma^0 E - \vec{\gamma}\vec{p} + m}{2E} \cdot e^{-iE(t-t') + i\vec{p}(\vec{x}-\vec{x}')}}$$

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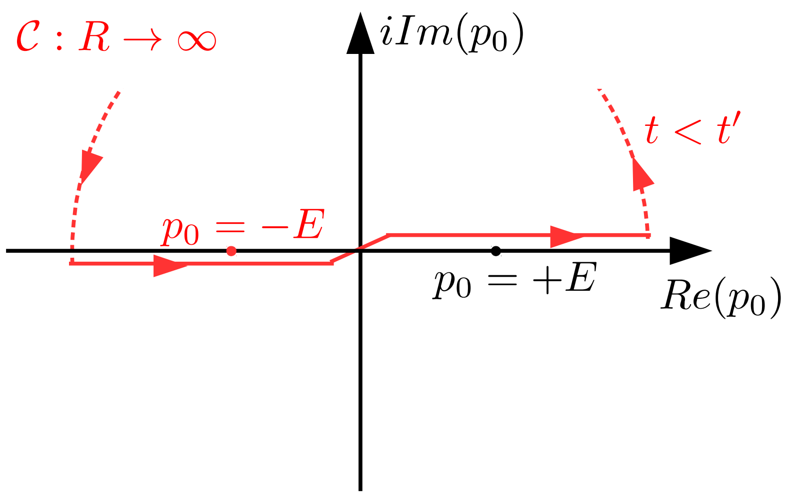

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- For $t < t'$ ($e^{+ip_0(t-t')} \to 0$ for $Im(p_0) \gg 0$):
  $\to$ close contour in upper plane & calculate integral from residual of enclosed pole.

$$\oint_{\mathcal{C}} \mathrm{d}p_0 \underbrace{\frac{1}{p_0 + E}}_{\text{pole at: } p_0 = -E} \cdot \underbrace{\frac{(\gamma^\mu p_\mu + m)}{p_0 - E} \, e^{-ip_0(t-t')}}_{\text{residuum: } f(p_0)} = +2\pi i \cdot f(p_0)\big|_{p_0 = E}$$

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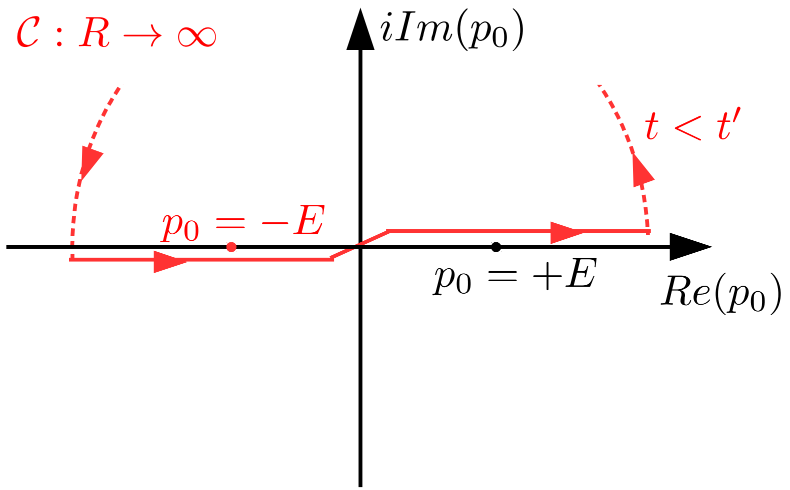
- Choose path $\mathcal{C}$ in complex plain to circumvent poles:



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$$K(x - x') = -i(2\pi)^{-3} \int \mathrm{d}^3\vec{p} \frac{-\gamma^0 E - \vec{\gamma}\vec{p} + m}{2E} \cdot e^{+iE(t-t') + i\vec{p}(\vec{x}-\vec{x}')}$$

The Fermion Propagator (Nota Bene)

- Choose path $\mathcal{C}$ in complex plain to circumvent poles:

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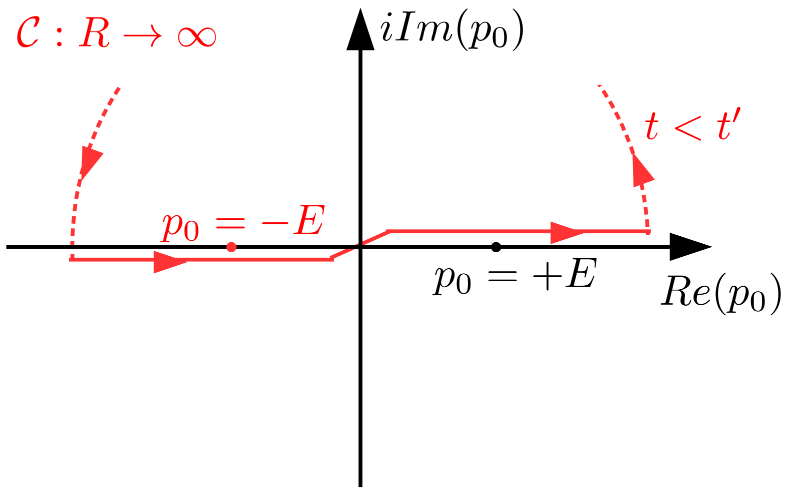

- The bending of the integration path can be circumvented by shifting the poles by ϵ.

$$\left[p_0 + \left(E - \frac{i\epsilon}{2E}\right)\right] \cdot \left[p_0 - \left(E - \frac{i\epsilon}{2E}\right)\right] = p_0^2 - \left(\vec{p}^{\,2} + m^2\right) + i\epsilon$$
$$= p^2 - m^2 + i\epsilon$$

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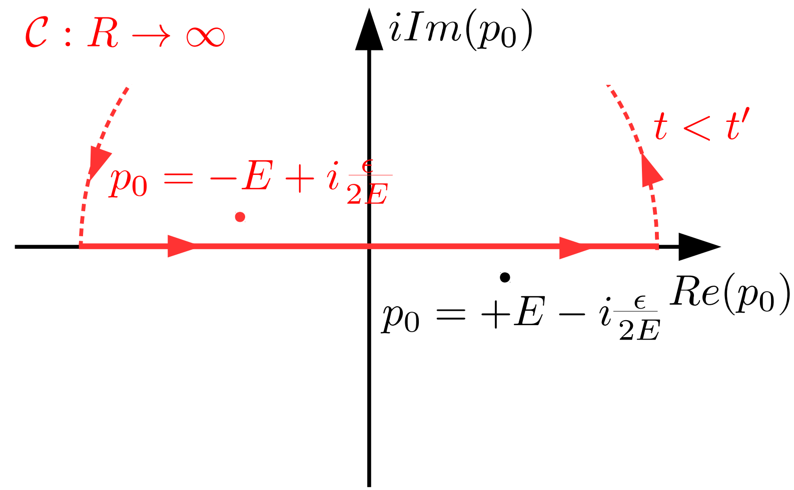

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$$\left[\underset{\downarrow \atop (-E + i\frac{\epsilon}{2E})}{p_0} + \left(E - \frac{i\epsilon}{2E}\right)\right] \cdot \left[\underset{\downarrow \atop (+E - i\frac{\epsilon}{2E})}{p_0} - \left(E - \frac{i\epsilon}{2E}\right)\right] = p_0^2 - \left(\vec{p}^{\,2} + m^2\right) + i\epsilon$$

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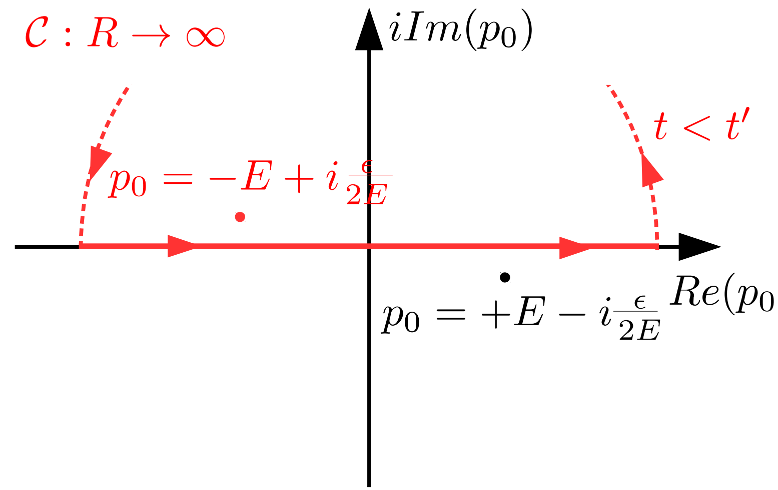

$$\int_{-\infty}^{+\infty} \mathrm{d}p_0 \, \frac{(\gamma^\mu p_\mu + m)}{(p_0 - E)(p_0 + E)} \, e^{-ip_0(t-t')}$$

- The bending of the integration path can be circumvented by shifting the poles by ϵ.

$$\left[p_0 + \left(E - \frac{i\epsilon}{2E}\right)\right] \cdot \left[p_0 - \left(E - \frac{i\epsilon}{2E}\right)\right] = p_0^2 - \left(\vec{p}^2 + m^2\right) + i\epsilon$$
$$= p^2 - m^2 + i\epsilon$$

$$\tilde{K}(p) = \frac{(\gamma^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon} \qquad \epsilon > 0$$

The Fermion Propagator (Summary & Time Development)

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- *Green's* function (for $t > t'$):

$$K(x - x') = -i(2\pi)^{-3} \int \mathrm{d}^3\vec{p}\, \frac{+\gamma^0 E - \vec{\gamma}\vec{p} + m}{2E} \cdot e^{-iE(t-t') + i\vec{p}(\vec{x}-\vec{x}')}$$

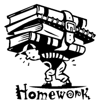
$$\phi(t, \vec{x}) = \begin{cases} i\int \mathrm{d}^3\vec{x}' K(x - x')\gamma^0 \phi(t', \vec{x}') & \text{for} \quad t > t' \\ 0 & \text{for} \quad t < t' \end{cases}$$

particle w/ pos. energy traveling forward in time.

$$\bar{\phi}(t, \vec{x}) = \begin{cases} 0 & \text{for} \quad t > t' \\ i\int \mathrm{d}^3\vec{x}' \bar{\phi}(t', \vec{x}')\gamma^0 K(x - x') & \text{for} \quad t < t' \end{cases}$$

particle w/ pos. energy traveling backward in time.

- Check the highlighted equation.



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Solution for ψ_{scat} (Perturbative Series)

- The integral equation can be solved iteratively:

$$\psi_{\text{scat}}(x) = -e \int K(x - x')\gamma^{\mu} A_{\mu}(x')\psi_{\text{scat}}(x')\mathrm{d}^4x'$$

  - 0th order perturbation theory:

    $\psi^{(0)}(x_f) = \phi_i(x_f)$ (solution of the homogeneous *Dirac* equation)

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This procedure is justified since e (in natural units) is small wrt. to 1:

$$\alpha = \frac{e^2}{4\pi\hbar c} \xrightarrow{\hbar = c = 1} \alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$$

  - 1st order perturbation theory:

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The Matrix Element $\mathcal{S}_{fi}$

- $\mathcal{S}_{fi}$ is obtained from the projection of the scattering wave ψ_{scat} on ϕ_f:

$$\begin{aligned}\mathcal{S}_{fi} = \int \mathrm{d}^3\vec{x}_f \phi_f^\dagger(x_f)\psi_{\text{scat}}(x_f) &= \int \mathrm{d}^3\vec{x}_f \phi_f^\dagger(x_f)\mathcal{S}\phi_i(x_f) \\ &= \delta_{fi} + \underbrace{\mathcal{S}_{fi}^{(1)}}_{\text{“LO”}} + \underbrace{\mathcal{S}_{fi}^{(2)}}_{\text{“NLO”}} + \ldots\end{aligned}$$

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cf. slide 33

$$\mathcal{S}_{fi}^{(1)} = i\cdot e\int \mathrm{d}^4x' \underbrace{\bar{\phi}_f(x')\gamma^\mu A_\mu(x')\phi_i(x')}_{\text{corresponds to the IA term in } \mathcal{L}\text{.}}$$

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- 1st order matrix element of the scattering amplitude.
- We still need to know A_μ.

The Photon Propagator

- Since the target particle is back scattered by the projectile, A_μ also evolves.
- This happens according to the inhomogeneous wave equation of the photon field (in *Lorentz* gauge $\partial_\mu A^\mu = 0$):

$$\boxed{\Box A^\mu = eJ^\mu}$$

- *Ansatz* via *Green's* function...:

$$A^\mu(x) = e\int \mathrm{d}^4x' D^{\mu\nu}(x-x')J_\nu(x')$$

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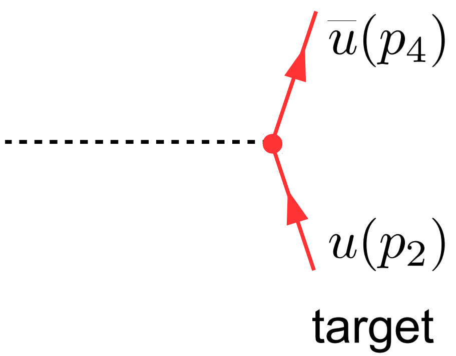
On the Way to Completion...

- *Ansatz* for target current:

$$\boxed{eJ^\mu(x) = e\cdot\overline{\psi}_f(x)\gamma^\mu\psi_i(x) = e\cdot\overline{u}(p_4)\gamma^\mu u(p_2)e^{i(p_4-p_2)x}}$$

$$\psi_i(x) = u(p_2)e^{-ip_2x} \qquad \psi_f(x) = u(p_4)e^{-ip_4x}$$



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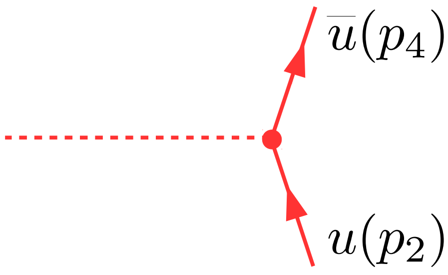

- Combination with photon propagator to get A_μ:

$$\begin{aligned} A^\mu(x) &= e\cdot\int \mathrm{d}^4x' \int \frac{\mathrm{d}^4q}{(2\pi)^4}\cdot\frac{-g^{\mu\nu}}{q^2+i\epsilon}e^{i(p_4-p_2+q)x'}e^{-iqx}\overline{u}(p_4)\gamma^\nu u(p_2) \\ &= e\cdot\int \mathrm{d}^4q\frac{-g^{\mu\nu}}{q^2+i\epsilon}\delta^4(p_4-p_2+q)e^{-iqx}\overline{u}(p_4)\gamma^\nu u(p_2) \end{aligned}$$

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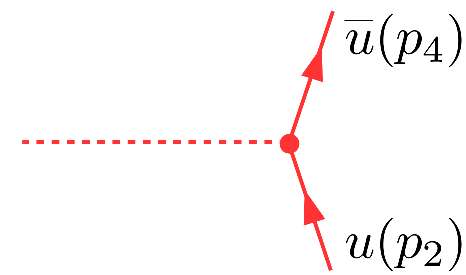

target

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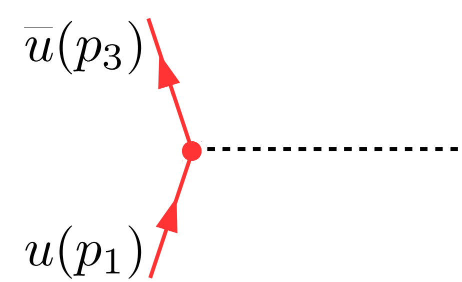
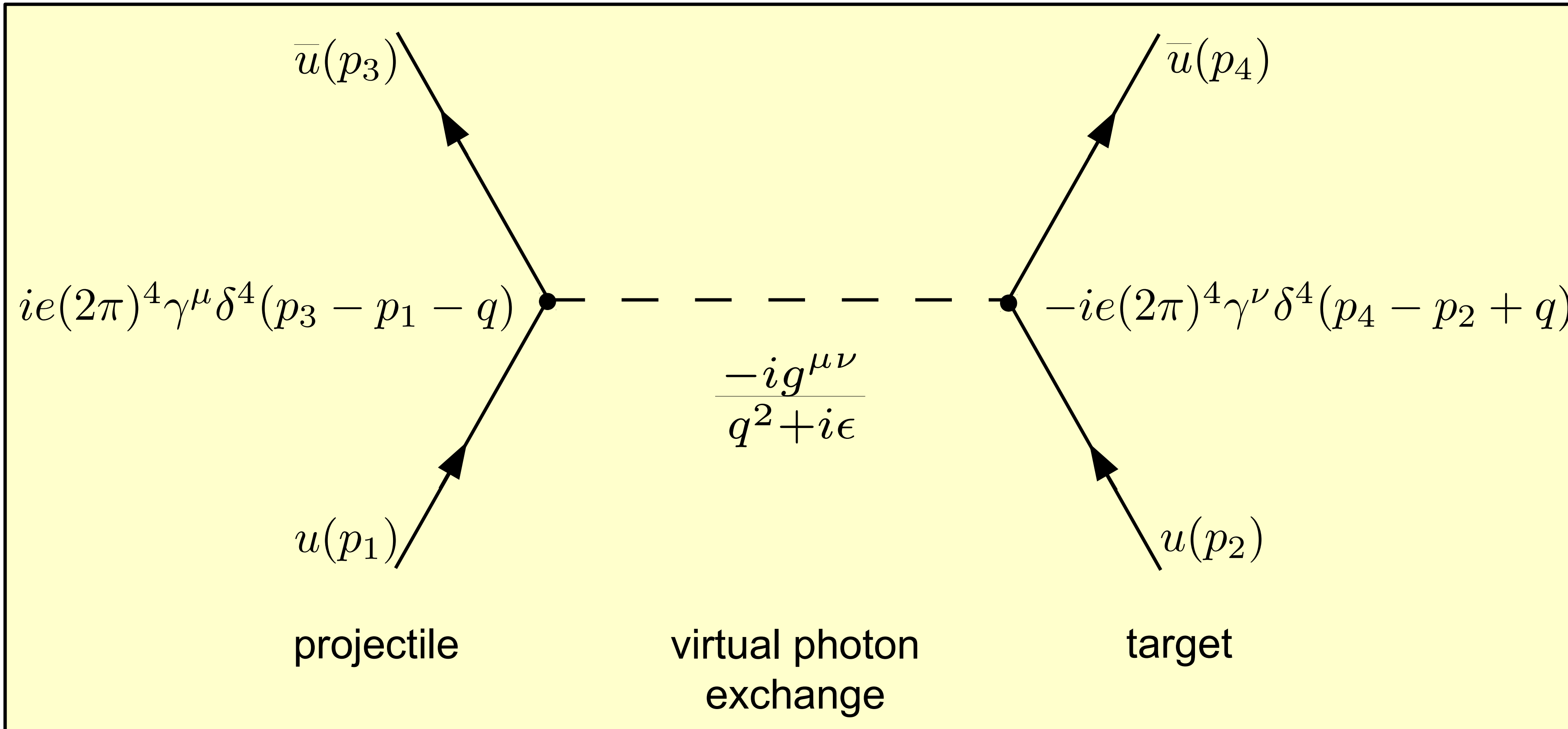


projectile

The Matrix Element $\mathcal{S}_{fi}$ (complete picture)

$$i\cdot e^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} (2\pi)^4 \overline{u}(p_3)\gamma_\mu u(p_1)\delta^4(p_3-p_1-q) \frac{-g^{\mu\nu}}{q^2+i\epsilon} \delta^4(p_4-p_2+q)(2\pi)^4\overline{u}(p_4)\gamma_\nu u(p_2)$$



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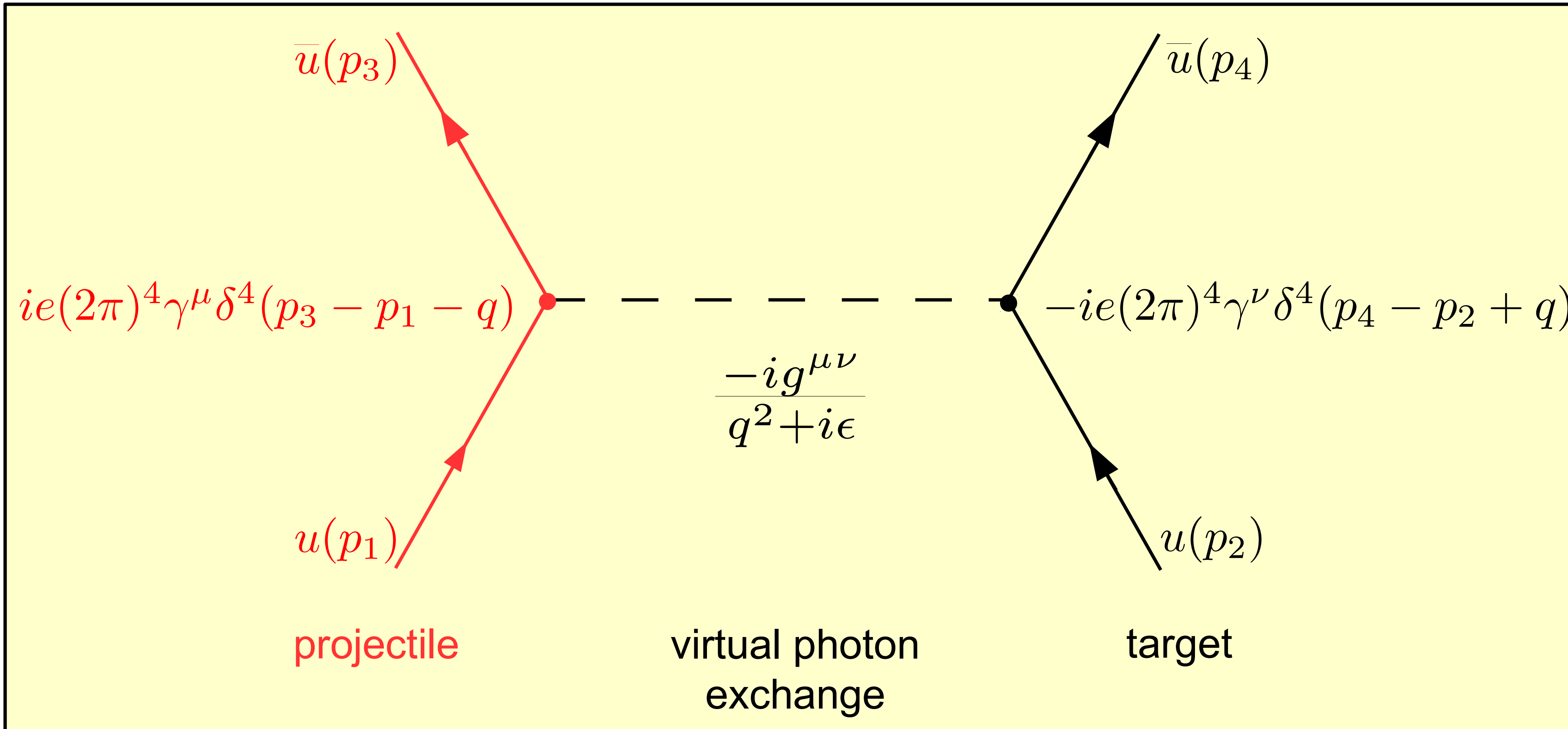

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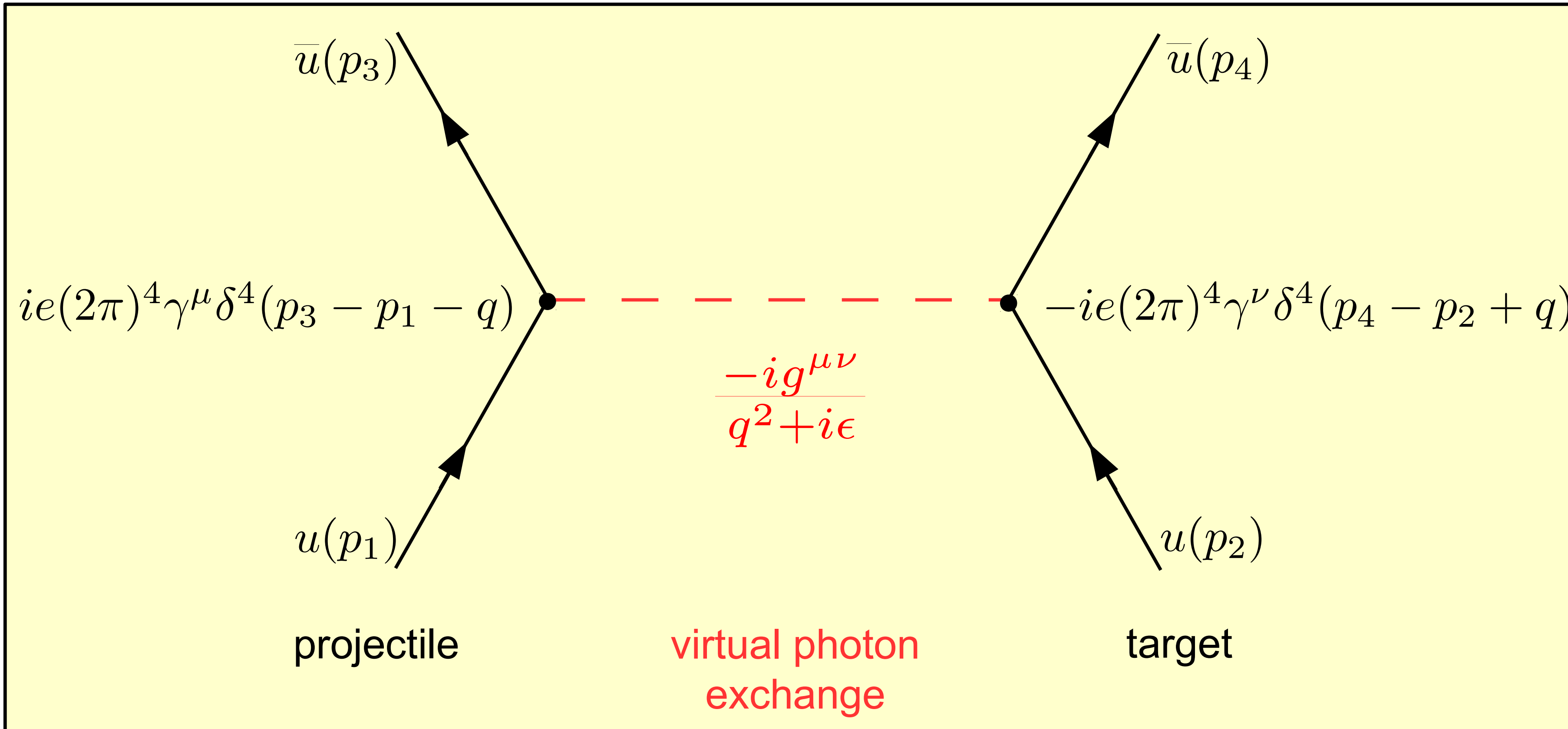

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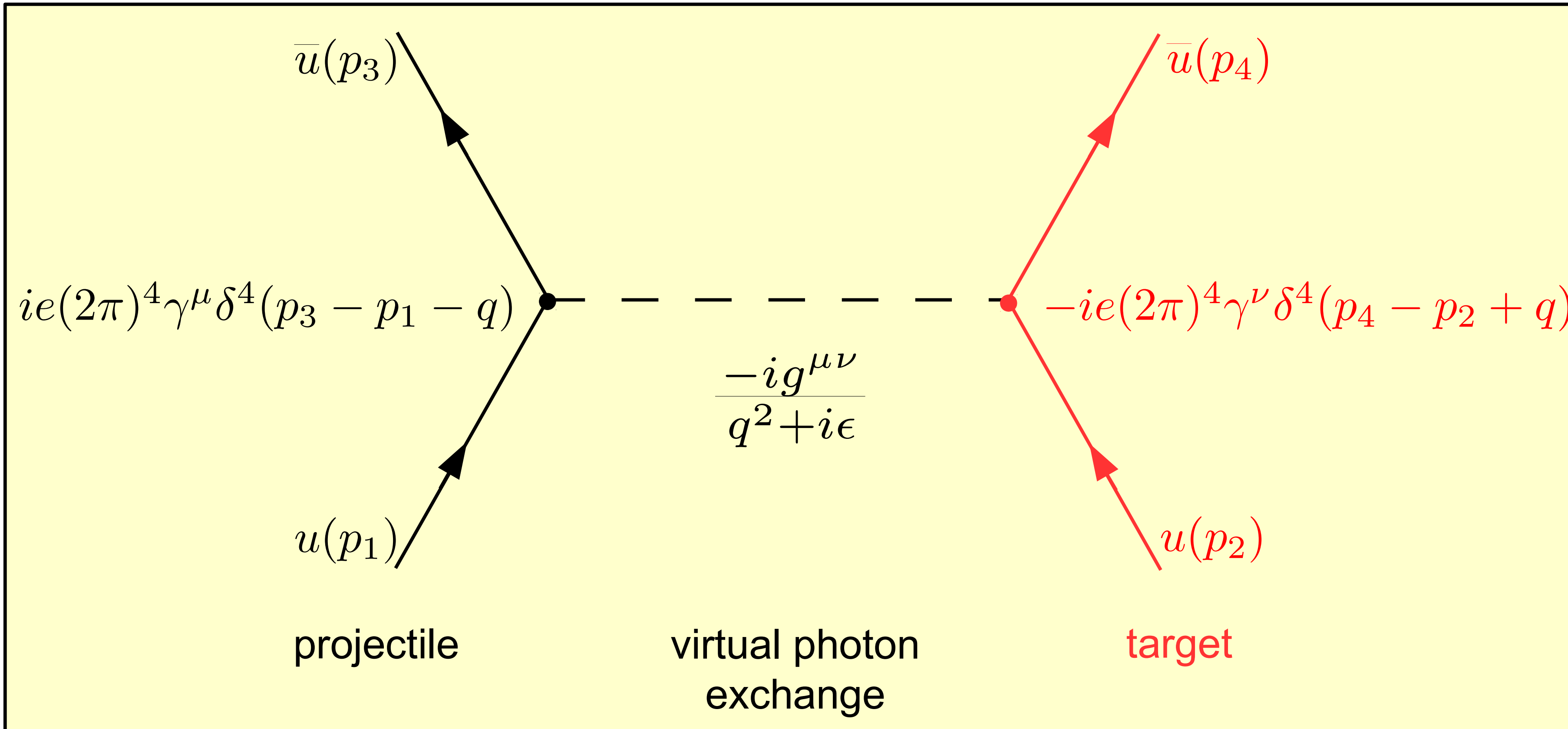

Feynman Rules (QED)

- *Feynman* diagrams are a way to represent the elements of the matrix element.
- The translation follows the *Feynman* rules:

Legs:		
	$u(p) \quad (\bar{u}(p))$	• Incoming (outgoing) lepton.
	$\epsilon_\mu(k) \quad (\epsilon^*_\mu(k))$	• Incoming (outgoing) photon.
Vertices:		
•	$-i(\pm e)\cdot(2\pi)^4\cdot\delta^4(p_f - p_i - q)$	• Lepton-photon vertex.
Propagators:		
	$\frac{i(\gamma^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon}$	• Incoming (outgoing) lepton.
	$\frac{-ig^{\mu\nu}}{q^2 + i\epsilon}$	• Incoming (outgoing) lepton.

- Four-momenta of all virtual particles have to be integrated out.

= they are not known...

Feynman Rules (QED)

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- *Feynman* diagrams are a way to represent the elements of the matrix element.
- A Feynman diagram:
  - is not a sketch, it is a mathematical representation!
  - is drawn in momentum space.
  - does not have a time direction. Only time information is introduced by choice of initial and final state by reader (e.g. t-channel vs s-channel processes).

Higher Order

Fixed Order Calculations

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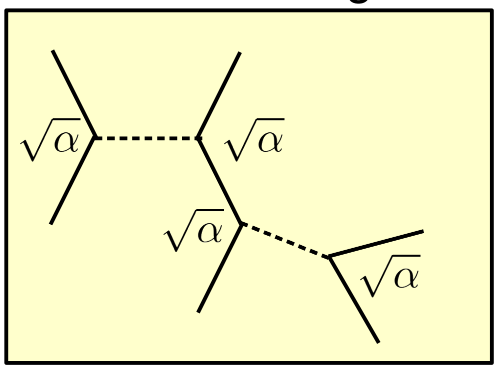
- Scattering amplitude $\mathcal{S}_{fi}$ is only known in perturbation theory!
- Works the better the smaller the perturbation is (= the coupling const.).
  - QED: $\alpha_{\text{em}} \approx \frac{1}{137}$
  - QFD: $\alpha_{\text{w}} = \alpha_{\text{em}} / \sin^2(\theta_W) \approx 4 \cdot \alpha_{\text{em}}$ $\theta_W = 28.74°$
  - QCD: $\alpha_s(m_Z) \approx 0.12$
- If perturbation theory works well, the first contribution of the scattering amplitude is already sufficient to describe the main features of the process.
- This contribution is of order "α". It is often called *Tree Level*, *Born Level* or *Leading Order* (LO) scattering amplitude.
- Any higher order of the scattering amplitude in perturbation theory appears at higher orders of "α".

Order α^2 Diagrams (QED)

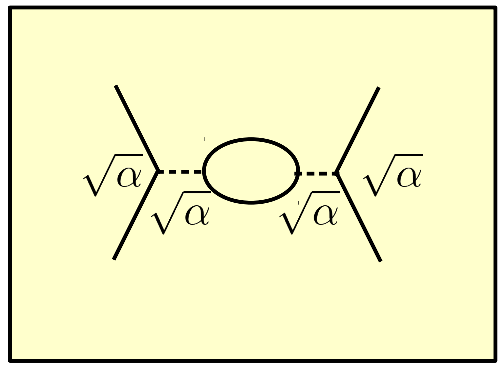
- We have only discussed contributions to $\mathcal{S}_{fi}$, which are of order α^1 in QED. (e.g. LO $ee \to ee$ scattering) .

- Diagrams which contribute to order α^2 would look like this:

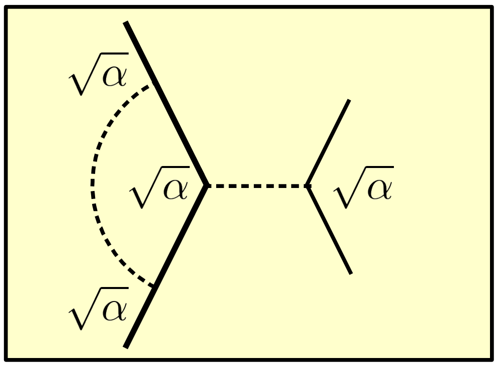
Additional legs:



Loops:



(loops in propagators or legs)



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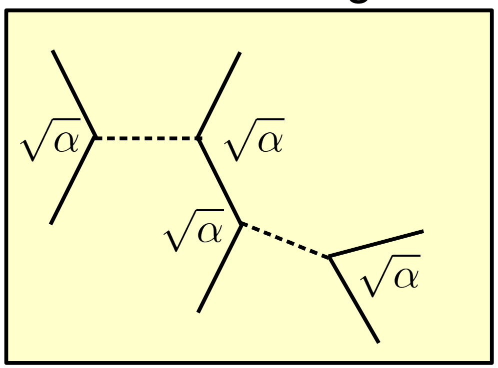

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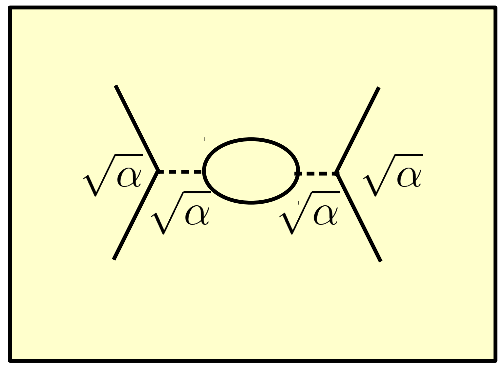

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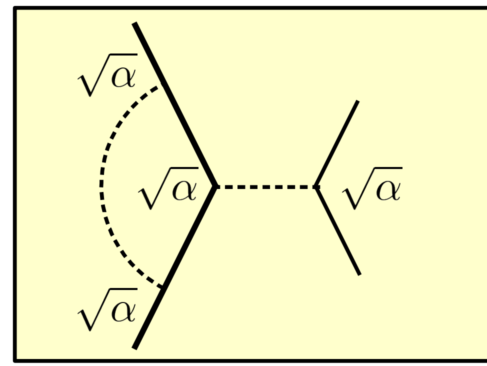

(loops in vertices)

- LO term for a $2 \to 4$ process.
- NLO contrib. for the $2 \to 2$ process.
- Opens phase spaces.

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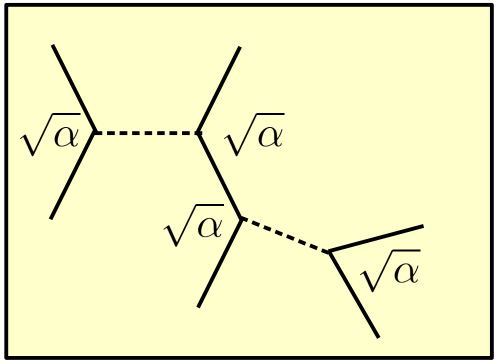

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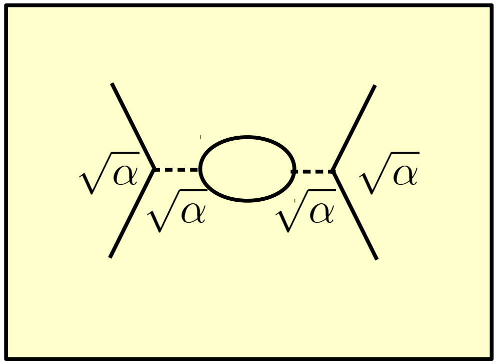
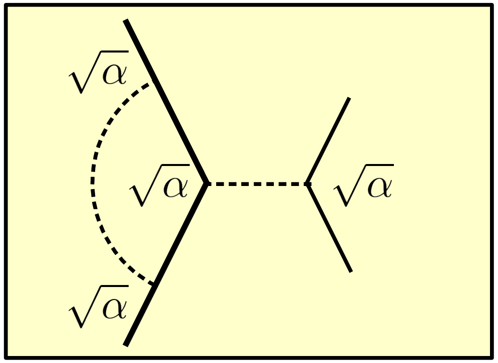


(loops in propagators or legs)

- Modify (effective) masses of particles ("running masses").



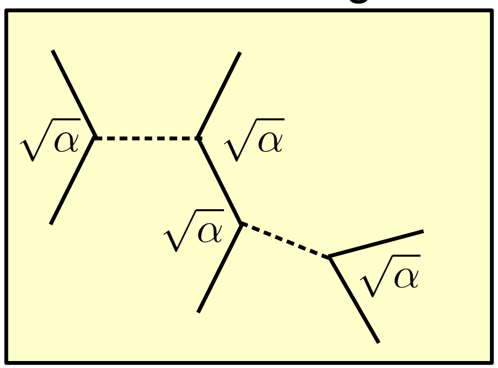
(loops in vertices)

Order α^2 Diagrams (QED)

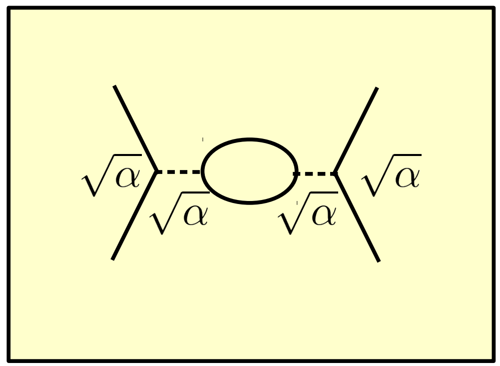
- We have only discussed contributions to $\mathcal{S}_{fi}$, which are of order α^1 in QED. (e.g. LO $ee \to ee$ scattering) .

- Diagrams which contribute to order α^2 would look like this:

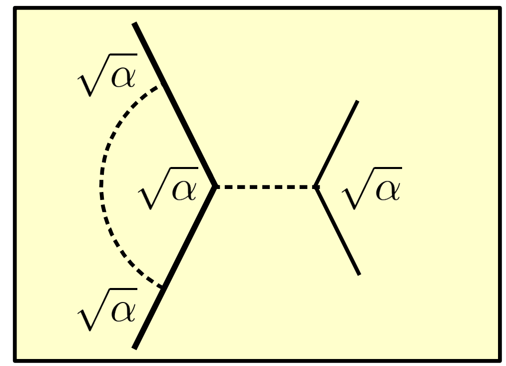
Additional legs:



Loops:



(loops in propagators or legs)



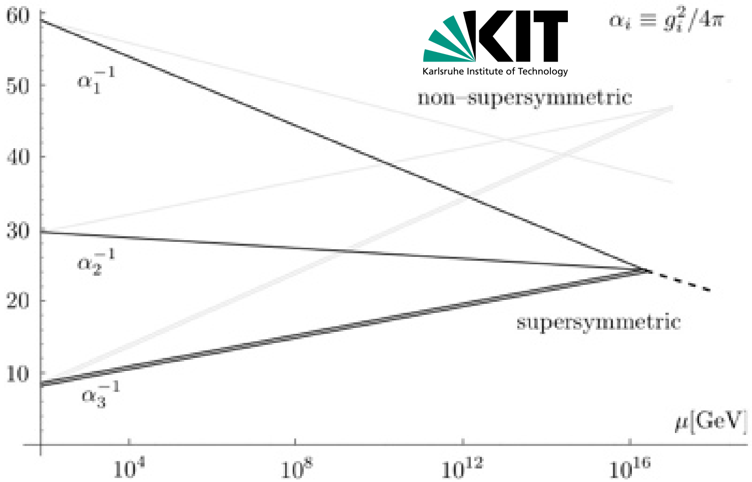
(loops in vertices)

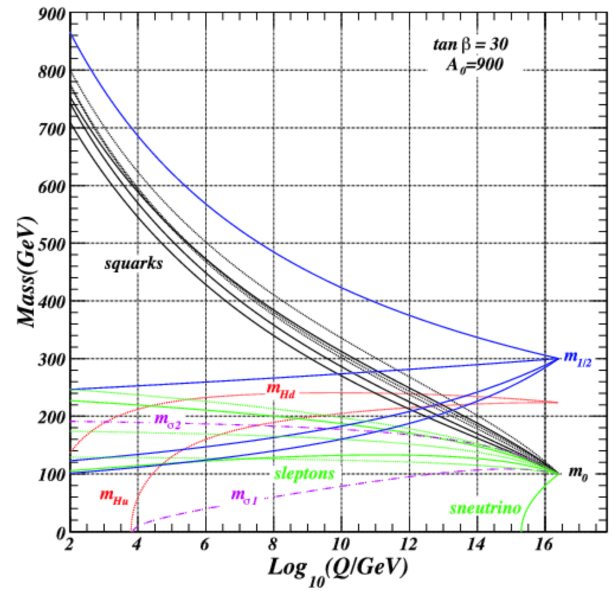
- LO term for a $2 \to 4$ process.
- NLO contrib. for the $2 \to 2$ process.
- Opens phase spaces.

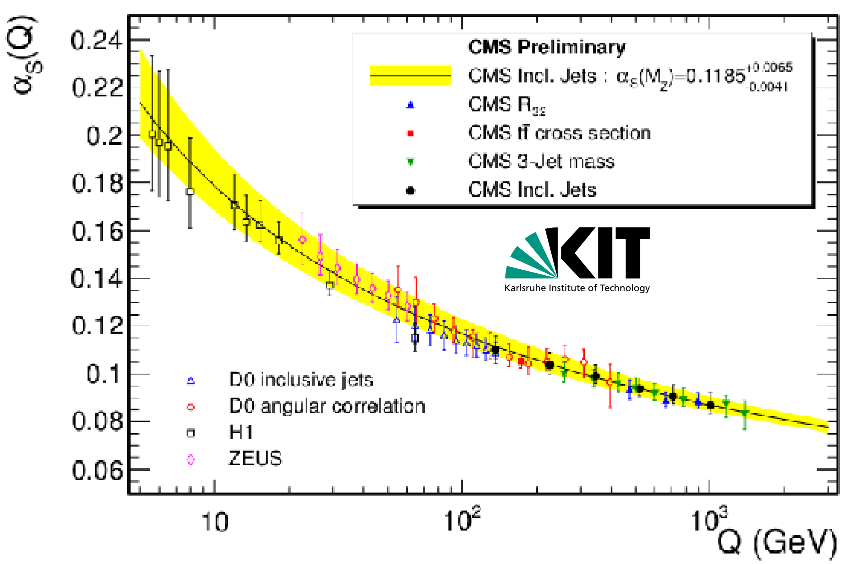
- Modify (effective) masses of particles ("running masses").

- Modify (effective) couplings of particles ("running couplings").

Examples for “Running Constants”







- Running of the constants can be predicted and is indeed observed.
- Coupling needs to be measured at least in one point.
- One usually gives the value at a reference scale (e.g. m_Z).

Effect of Higher Order Corrections

- Change over all normalization of cross sections (e.g. via change of coupling, but also by kinematic opening of phase space – large effect)
- Change kinematic distributions (e.g. harder or softer transverse momentum spectrum of particles)
- In QED effects are usually “small” (correction to LO is already at $O(1\%)$ level). In QCD effects are usually “large” ($O(10\%)$). Therefore reliable QCD predictions almost always require (N)NLO.
- Higher orders can be mixed (e.g. $O(\alpha\alpha_s^2)$).
- In concrete calculations the number of contributing diagrams quickly explodes for higher order calculations, which makes these calculations very difficult.

Boundaries on the Higgs Mass within the SM

The Running of λ in the Higgs Potential

- Like the couplings α_{em} , α_{w} and α_{s} also the self-coupling λ in the Higgs potential is subject to higher order corrections:

$$\mathcal{L}^{\mathrm{Higgs}} = \partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi)$$

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda \left(\phi^\dagger \phi\right)^2$$

(Higgs potential)

$$\boxed{\frac{\mathrm{d}\lambda}{\mathrm{d}\log Q^2} = \frac{1}{16\pi^2}\left[\underbrace{12\lambda^2}_{\text{Higgs}} + \underbrace{6\lambda f_t^2 - 3f_t^4}_{\textit{top}\text{ quark}} - \frac{3}{2}\lambda\left(3\alpha_{\mathrm{em}}^2 + \alpha_{\mathrm{w}}^2\right) + \ldots\right]}$$

(Renormalization group equation at 1-loop accuracy)

- Since the Higgs boson couples proportional to the mass the high energy behavior of λ will be dominated by the heaviest object in the loop.

The Running of λ in the Higgs Potential

- First case: large Higgs mass ($m_H \gg Q^2$)

$$\boxed{\frac{\mathrm{d}\lambda}{\mathrm{d}\log Q^2} = \frac{1}{16\pi^2}\left[\underbrace{12\lambda^2}_{\text{Higgs}} + \underbrace{6\lambda f_t^2 - 3f_t^4}_{\textit{top}\text{ quark}} - \frac{3}{2}\lambda\left(3\alpha_{\mathrm{em}}^2 + \alpha_{\mathrm{w}}^2\right) + \ldots\right]}$$

$$\Big\downarrow\; m_H \gg Q^2$$

$$\boxed{\frac{\mathrm{d}\lambda}{\mathrm{d}\log Q^2} = \frac{3}{4\pi^2}\lambda^2(Q^2)} \xrightarrow{\text{solution}} \lambda(Q^2) = \frac{\lambda(\mathrm{v}^2)}{1 - \frac{3}{4\pi^2}\lambda(\mathrm{v}^2)\log\left(Q^2/\mathrm{v}^2\right)}$$

(vacuum expectation value: $\mathrm{v}^2 = \mu^2/\lambda$)

- For $Q^2 \ll \mathrm{v}^2 = 246\,\mathrm{GeV}$ we get $\log\left(Q^2/\mathrm{v}^2\right) \ll 0$ and $\lambda(Q^2) \to 0$.
- For increasing Q^2 $\lambda(Q^2)$ will run into a pole and become non-perturbative!

The Running of λ in the Higgs Potential

- First case: large Higgs mass ($m_H \gg Q^2$)

$$\boxed{\frac{\mathrm{d}\lambda}{\mathrm{d}\log Q^2} = \frac{1}{16\pi^2}\left[\underbrace{12\lambda^2}_{\text{Higgs}} + 6\lambda f_t^2 \underbrace{- 3f_t^4 - \frac{3}{2}\lambda\left(3\alpha_{\mathrm{em}}^2 + \alpha_{\mathrm{w}}^2\right)}_{\textit{top}\text{ quark}} + \ldots\right]}$$

$$\Big\downarrow \; m_H \gg Q^2$$

$$\boxed{\frac{\mathrm{d}\lambda}{\mathrm{d}\log Q^2} = \frac{3}{4\pi^2}\lambda^2(Q^2)} \xrightarrow{\text{solution}} \lambda(Q^2) = \frac{\lambda(\mathrm{v}^2)}{1 - \frac{3}{4\pi^2}\lambda(\mathrm{v}^2)\log\left(Q^2/\mathrm{v}^2\right)}$$

(vacuum expectation value: $\mathrm{v}^2 = \mu^2/\lambda$)

- From this (*Landau*) pole an upper bound can be derived on $m_H = \mu$, depending on up to which scale the theory should remain perturbative.

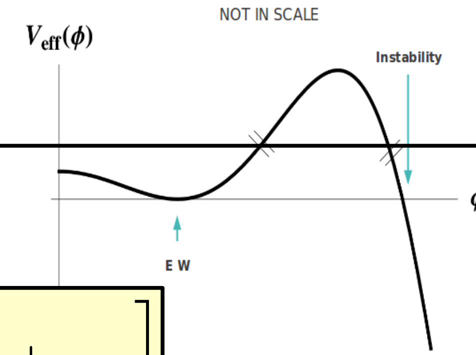
Intrinsic Bounds on m_H

- The upper bound on m_H due to the *Landau* pole is called *triviality bound*:

$$m_H\left(Q(\text{Landau}) = 10^{3}\,\text{GeV}\right) \leq 1.6\,\text{TeV}$$
$$m_H\left(Q(\text{Landau}) = 10^{16}\,\text{GeV}\right) \leq 340\,\text{GeV}$$

(Triviality bound)

The Running of λ in the Higgs Potential



- Second case: small Higgs mass ($m_H \ll m_t$)

$$\boxed{\frac{\mathrm{d}\lambda}{\mathrm{d}\log Q^2} = \frac{1}{16\pi^2}\left[\underbrace{12\lambda^2}_{\text{Higgs}} + \underbrace{6\lambda f_t^2 - 3f_t^4}_{\textit{top}\text{ quark}} - \frac{3}{2}\lambda\left(3\alpha_{\mathrm{em}}^2 + \alpha_{\mathrm{w}}^2\right) + \ldots\right]}$$

$$\Big\downarrow\ m_H \ll m_t$$

$$\boxed{\frac{\mathrm{d}\lambda}{\mathrm{d}\log Q^2} = -\frac{3}{16\pi^2} f_t^4} \xrightarrow{\text{solution}} \lambda(Q^2) = \lambda(\mathrm{v}^2) - \frac{3}{16\pi^2}\frac{m_t^4}{\mathrm{v}^4}\log\left(Q^2/\mathrm{v}^2\right)$$

(with: $f_t = m_t/\mathrm{v}$)

- With $\lambda(\mathrm{v}^2) = \mu^2/\mathrm{v}^2$ and increasing Q^2 $\lambda(Q^2)$ will turn negative and the Higgs potential will no longer be bound from below. The vacuum turns instable.

Intrinsic Bounds on m_H

- The upper bound on m_H due to the *Landau* pole is called *triviality bound*:

$$m_H\left(Q(\mathrm{Landau}) = 10^{3}\,\mathrm{GeV}\right) \leq 1.6\,\mathrm{TeV}$$

$$m_H\left(Q(\mathrm{Landau}) = 10^{16}\,\mathrm{GeV}\right) \leq 340\,\mathrm{GeV}$$

(Triviality bound)

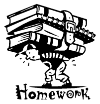
- The lower bound on m_H is called *stability bound*:

$$m_H\left(Q(\mathrm{Landau}) = 10^{3}\,\mathrm{GeV}\right) \geq 20\,\mathrm{GeV}$$

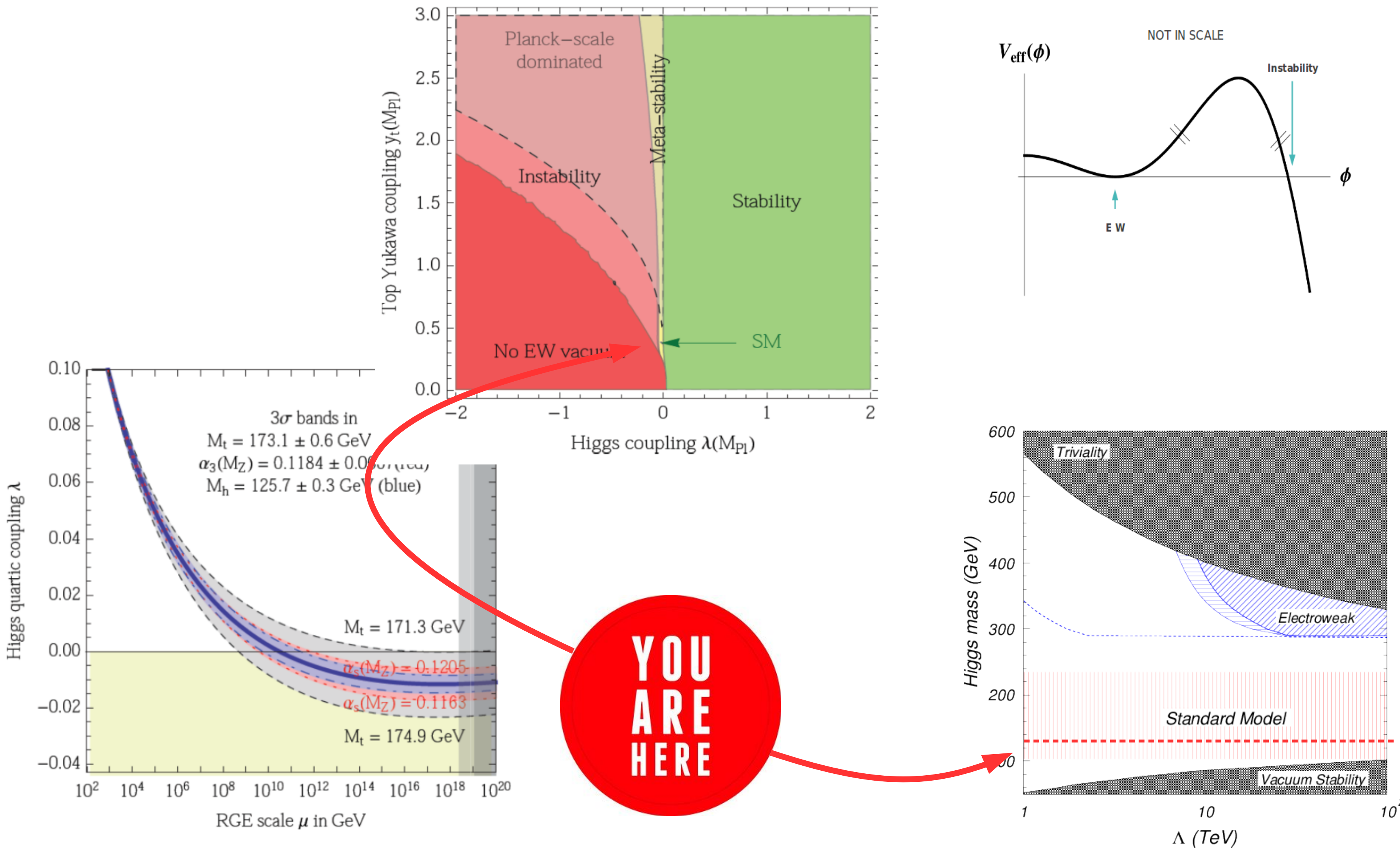
$$m_H\left(Q(\mathrm{Landau}) = 10^{16}\,\mathrm{GeV}\right) \geq 90\,\mathrm{GeV}$$

(Stability bound)

- Calculate the boundaries from the equations that have been given.



Intrinsic Bounds on m_H



Concluding Remarks

- The amplitude of scattering processes can be obtained from a QM model via perturbation theory.
- We have derived the propagators as formal solutions of the equations of motion for the photon and for the electron.
- We have contracted the propagators and the fermion *spinors* into the matrix element to obtain its final form.
- We have reviewed the *Feynman* rules to translate the matrix element into a pictorial form and discussed the effect of higher order corrections.
- Finally we have seen how higher order corrections within the model give boundaries on the mass of the Higgs boson already within the model from requirements on its applicability.

Sneak Preview for Next Week(s)

- Next week Günter Quast will take over for the next lecture (21. Mai!).

- You will discuss the way from observable to measurement:
  - Rate measurements and measurements of particle properties.
  - Monte Carlo methods for event simulation.
  - Parton showers, hadronization, detector simulation.

- The week after you will discuss basic experimental measurement techniques (this will be done by Andrew Gilbert, 28. Mai):
  - Data acquisition, triggers.
  - Event selections, object calibration, reconstruction efficiencies, acceptances.
  - Determination of background processes.

Backup & Homework Solutions

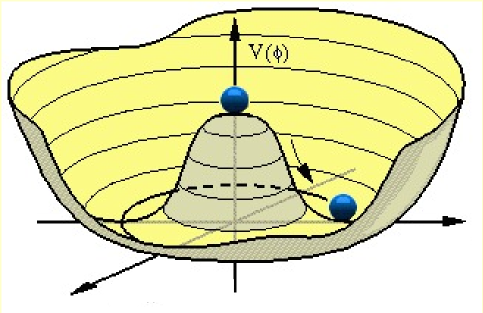
Recap from Last Time

- Introduced new field ϕ as $SU(2)$ doublet in the theory:

$$\mathcal{L}^{SU(2)\times U(1)} = \mathcal{L}^{\text{kin}} + \mathcal{L}^{CC} + \mathcal{L}^{NC} + \mathcal{L}^{\text{gauge}} + \mathcal{L}^{\text{Higgs}}$$

$$\mathcal{L}^{\text{Higgs}} = \partial_\mu \phi^+ \partial^\mu \phi - V(\phi)$$

$$V(\phi) = -\mu^2 \phi^+ \phi + \lambda \left(\phi^+ \phi\right)^2$$



- Coupled ϕ to $SU(2)$ gauge fields (via covariant derivate).

- Developed ϕ in its energy ground state and obtained massive gauge bosons, massive Higgs boson and massive fermions via coupling to ϕ :
  - Higgs boson obtains mass via *Goldstone* potential.
  - Gauge bosons obtain mass via gauge invariance requirement ($\rightarrow$ covariant derivative).
  - Fermions obtain mass via “conventional” *Yukawa* coupling to ϕ.