Teilchenphysik II (Higgs-Physik) (SS 2016)

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Exercises Sheet 02

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Exercise 3: Dirac Equation from Lagrangian Density (presence)

The Lagrangian density is defined via the action integral as

$$\mathcal{L}: \qquad \mathcal{S} = \int_\Omega \mathcal{L}(\{\partial_\mu \phi_i\}, \{\phi_i\}) \mathrm{d}t\mathrm{d}^3x\,.$$

What is the dimension of the Lagrangian density in natural units?

Exercise 4: Gauge Transformation of A_μ (presence)

As discussed in the lecture the covariant derivate is defined by the replacement:

$$\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$$

with the additional gauge field A_μ and the coupling constant e. The transformation behavior of the covariant derivative is given by:

$$D_\mu \to D'_\mu = D_\mu - i\partial_\mu \vartheta$$

where $\vartheta = \vartheta(x)$ is a local phase change of the external fields. What is the transformation behavior of the gauge field A_μ?

Exercise 5: Dirac Equation from Lagrangian Density (presence)

In the lecture we have derived the *Klein-Gordon* equation from the corresponding Lagrangian density for free bosons

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi\phi^*$$

using the *Euler-Lagrange* equations applied to the field ϕ^*. Do the same exercise to obtain the *Dirac* equation by applying the *Euler-Lagrange* equations to the field ψ for the Lagrangian density for free fermions:

$$\mathcal{L} = i\overline{\psi}\gamma^\mu \partial_\mu \psi - \overline{\psi} m \psi\,.$$

Exercise 6: Gauge Invariance of $F_{\mu\nu}$ (presence)

In the lecture we have made the *ansatz*

$$\mathcal{L}_{kin} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \qquad \text{with} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

for the kinetic term of the gauge field in the full Lagrangian density. Using the translation behavior of the gauge field as obtained in Exercise 4. Proof that $F'_{\mu\nu} = F_{\mu\nu}$. As a consequence the term $\mathcal{L}_{kin}$ is not only manifest Lorentz invariant, but also gauge invariant (i.e. it fulfills the required transformation behavior of the Lagrangian density).

Exercise 7: Local Gauge Invariance for Bosons (homework)

In the lecture we have sketched the exercise to enforce local gauge invariance starting from the Lagrangian density for free propagating fermions. We have seen how this leads to the full Lagrangian density of Quantum Electrodynamics (QED). In nature we also have charged bosons, for which the same procedure should work. How do you know that this is true?

a) Proof that the same covariant derivative with the same gauge transformation laws works equally well for bosons as for fermions. Translate the transformation behavior for fermions to bosons

$$\begin{aligned}\phi(\vec{x},t) &\to \phi'(\vec{x},t) = e^{i\theta}\phi(\vec{x},t)\\ \phi^*(\vec{x},t) &\to \phi'^*(\vec{x},t) = \phi^*(\vec{x},t)e^{-i\theta}\\ D_\mu &\to D'_\mu \quad = D_\mu - i\partial_\mu\theta\end{aligned}$$

apply it to the Lagrangian density term for bosons and proof the relation

$$\mathcal{L}' = D'_\mu\phi'\left(D'^\mu\phi'\right)^* - m^2\phi'\phi'^* = D_\mu\phi\left(D^\mu\phi\right)^* - m^2\phi\phi^* = \mathcal{L}\,.$$

b) Write out the full Lagrangian density term for bosons in analogy to the Lagrangian density term $\mathcal{L}_{QED}$ that has been given for fermions in the lecture. (You can add the term for the free gauge field for completeness, if you like, but this is not important for the point that we want to make here.) Derive the equations of motion starting from this Lagrangian density term and compare it with the fermion case that you have seen in the lecture.

Exercise 8: Variation of the Free Gauge Field A_μ (homework)

In the lecture we have shown how from the variation of the free gauge field term

$$\mathcal{L}_{kin} = \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

the *Klein-Gordon* equation for a free massless boson follows, which can be shown in the physical *Lorentz* gauge of electrodynamics. Try to follow the line of arguments step by step starting from the *Euler-Lagrange* equations:

$$\partial_\mu\frac{\partial\mathcal{L}_{kin}}{\partial(\partial_\mu A_\nu)} - \frac{\partial\mathcal{L}_{kin}}{\partial A_\nu} = 0\,.$$

Especially proof the (non-trivial) missing piece that we have not shown in the lecture:

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = \partial_\mu F^{\mu\nu}\,.$$