Teilchenphysik II (Higgs-Physik) (SS 2016)

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Exercises Sheet 06

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Exercise 21: *Green*'s Function of the *Dirac* Equation (presence)

In the lecture we have introduced the *Green*'s function $K(x-x')$ for solving of the inhomogeneous *Dirac* equation:

$$(i\gamma^\mu \partial_\mu - m)\,\psi = -e\gamma^\mu A_\mu \psi\,. \tag{1}$$

where $K(x-x')$ has the property:

$$(i\gamma^\mu \partial_\mu - m)\,K(x-x') = \delta^4(x-x') \tag{2}$$

and $\delta^4(x-x')$ is the four dimensional delta distribution. Proof that even without knowing the exact form of $K(x-x')$, just due to the property of Equation (2) we know that

$$\psi(x) = -e\int K(x-x')\gamma^\mu A_\mu(x')\psi(x')\mathrm{d}^4x' \tag{3}$$

is a solution of Equation (1) in the point x if $\psi(x')$ is known in point x'. Note that Equation (3) is not the solution of Equation (1). But it turns the differential equation into an integral equation.

Exercise 22: *Fourier* Transform of the *Green*'s Function (presence)

Find the concrete form of the *Fourier* transform

$$K(x-x') = (2\pi)^{-4}\int \tilde{K}(p)e^{-ip(x-x')}\mathrm{d}^4p$$

of the *Green*'s function. Make use of the fact that

$$\delta^4(x-x') \equiv (2\pi)^{-4}\int \mathbb{I}_4 e^{-ip(x-x')}\mathrm{d}^4p$$

$K(x-x')$ and $\tilde{K}(p)$ are space and momentum space representations of the fermion propagator. The fermion propagator is a 4×4 matrix that acts in the *spinor* space. It is only defined for virtual fermions.

Exercise 23: Concrete Solution of the Inhomogeneous *Dirac* Equation (presence)

In the lecture we have derived the *Green*'s function for a forward propagating field with positive energy (for $p_0 = E > 0$ and $t > t'$):

$$K(x-x') = -i(2\pi)^{-3}\int \mathrm{d}^3\vec{p}\,\frac{+\gamma^0E - \vec{\gamma}\vec{p} + m}{2E}\cdot e^{-iE(t-t')+i\vec{p}(\vec{x}-\vec{x'})} \tag{4}$$

We have implemented this boundary condition for the solution of Equation (1) by our choice of the integration path. Show explicitly that the solution $\phi(t,\vec{x})$ of Equation (1) does indeed have the desired time evolution behavior:

$$\phi(t,\vec{x}) = \begin{cases} i\int \mathrm{d}^3\vec{x}'K(x-x')\gamma^0\phi(t',\vec{x}') & \text{for}\, t > t' \\ 0 & \text{for}\, t < t' \end{cases}$$

For this make the *ansatz* $\phi(t',\vec{x}') = u(k)e^{-ik_0t'+i\vec{k}\vec{x}'}$ for an undisturbed plane wave at $(t',\vec{x}')$ and evolve it from $(t',\vec{x}')$ to $(t,\vec{x})$.

Exercise 21: *Green*'s Function of the *Dirac* Equation (solution)

We set Equation (3) into the right hand side of Equation (1):

$$(i\gamma^\mu\partial_\mu - m)\,\psi(x) = -e\int \underbrace{(i\gamma^\mu\partial_\mu - m)\,K(x-x')}_{\delta^4(x-x')}\gamma^\mu A_\mu(x')\psi(x')\mathrm{d}^4x'$$

Integration over x' leads to

$$(i\gamma^\mu\partial_\mu - m)\,\psi(x) = -e\gamma^\mu A_\mu(x)\psi(x)$$

which corresponds ot the right hand side of Equation (3).

Exercise 22: *Fourier* Transform of the *Green*'s Function (solution)

We know the following relations:

$$\begin{aligned}(i\gamma^\mu\partial_\mu - m)\,K(x-x') &= \delta^4(x-x') \qquad \text{(Equation (2))}\\ (i\gamma^\mu\partial_\mu - m)\,K(x-x') &= (2\pi)^{-4}\int (\gamma^\mu p_\mu - m)\,\tilde{K}(p)e^{-ip(x-x')}\mathrm{d}^4p\end{aligned}$$

Further-on we can make use of the knowledge of the *Fourier* transform of the delta distribution

$$\delta^4(x-x') \equiv (2\pi)^{-4}\int \mathbb{I}_4 e^{-ip(x-x')}\mathrm{d}^4p$$

From the uniqueness of the *Fourier* transformation we can conclude that

$$(\gamma^\mu p_\mu - m)\,\tilde{K}(p) = \mathbb{I}_4$$

Keep in mind that also the left hand side of this equation is a 4×4 matrix in the *spinor* space. The inverted matrix can be obtained in an elegant way by multiplication with $(\gamma^\mu p_\mu + m)$ from the right:

$$(\gamma^\mu p_\mu + m)\cdot(\gamma^\mu p_\mu - m)\,\tilde{K}(p) = (\gamma^\mu p_\mu + m)\cdot\mathbb{I}_4$$

From this we obtain a form which is close to the final form that we know for the fermion propagator:

$$\tilde{K}(p) = \frac{(\gamma^\mu p_\mu + m)}{p^2 - m^2}$$

Exercise 23: *Fourier* Transform of the *Green*'s Function (solution)

We make use of the explicit form of the *Green*'s function that has been derived in the lecture:

$$K(x-x') = -i(2\pi)^{-3}\int \mathrm{d}^3\vec{p}\,\frac{+\gamma^0 E - \vec{\gamma}\vec{p} + m}{2E}\cdot e^{-iE(t-t')+i\vec{p}(\vec{x}-\vec{x}')} \qquad \text{(Equation (4))}$$

Further-on we make the *ansatz*

$$\phi(t',\vec{x}') = u(k)e^{-ik_0t'+i\vec{k}\vec{x}'}$$

for an undisturbed plane wave at $(t',\vec{x}')$ that we evolve from $(t',\vec{x}')$ to $(t,\vec{x})$. We set both into the general solution of Equation (1) and obtain:

$$\begin{aligned}
\phi(t,x) &= i\int \mathrm{d}\vec{x}' K(x-x')\gamma^0\phi(t',\vec{x}') \\
&= i\int \mathrm{d}\vec{x}' \underbrace{-i(2\pi)^{-3}\int \mathrm{d}^3\vec{p}\,\frac{+\gamma^0 E - \vec{\gamma}\vec{p} + m}{2E}\cdot e^{-iE(t-t')+i\vec{p}(\vec{x}-\vec{x}')}}_{K(x-x')\,(t>t')}\gamma^0\underbrace{u(k)e^{-ik_0t'+i\vec{k}\vec{x}'}}_{\phi(t',\vec{x}')} \\
&= -i^2\int \mathrm{d}^3\vec{p}\,\underbrace{(2\pi)^{-3}\int \mathrm{d}\vec{x}'\,e^{i(\vec{k}-\vec{p})\vec{x}'}}_{\delta^3(\vec{k}-\vec{p})}\frac{\left(+\gamma^0 E - \vec{\gamma}\vec{p} + m\right)\gamma^0 u(k)}{2E}\cdot e^{-iEt+i\vec{p}\vec{x}}\cdot e^{-i(k_0-E)t'} \\
&= \frac{\left(\gamma^0 k_0 - \vec{\gamma}\vec{k} + m\right)\gamma^0 u(k)}{2k_0}\cdot e^{-ik_0t+i\vec{k}\vec{x}} \\
&= \frac{\gamma^0\left(\gamma^0 k_0 + \vec{\gamma}\vec{k} + m\right)u(k)}{2k_0}\cdot e^{-ik_0t+i\vec{k}\vec{x}} \\
&= u(k)\cdot e^{-ik_0t+i\vec{k}\vec{x}}
\end{aligned}$$

which is, as desired the plane wave at (t,x). Note the replacement of $E\to k_0$ due to the evaluation of $\delta^3(\vec{k}-\vec{p})$ which in consequence also leads to $e^{-i(k_0-E)t'}\equiv 0$ in line three of the equation. In line five of the equation we have swapped γ^0 with the term in braces (with the consequence of a sign flip for $\vec{\gamma}\vec{k}$ in braces). In the last line of the equation we have made use of the *Dirac* equation for the free particle solution in the initial state:

$$\begin{aligned}
\left(\gamma^0 k_0 - \vec{\gamma}\vec{k} - m\right)u(k) &= 0 \\
\left(\vec{\gamma}\vec{k} + m\right)u(k) &= \gamma^0 k_0 u(k)
\end{aligned}$$

The calculation for $t<t'$ is only different in the use of the "backward propagator" instead of the "forward propagator" of $K(x-x')$ to resolve the integral:

$$K(x-x') = -i(2\pi)^{-3} \int \mathrm{d}^3\vec{p}\, \frac{-\gamma^0 E - \vec{\gamma}\vec{p} + m}{2E} \cdot e^{+iE(t-t')+i\vec{p}(\vec{x}-\vec{x}')}$$

This has a few sign flips in the above equation as consequence:

$$\phi(t,x) = i \int \mathrm{d}\vec{x'} K(x-x')\gamma^0 \phi(t',\vec{x'})$$

$$= i \int \mathrm{d}\vec{x'}\, \underbrace{-i(2\pi)^{-3} \int \mathrm{d}^3\vec{p}\, \frac{-\gamma^0 E - \vec{\gamma}\vec{p} + m}{2E} \cdot e^{+iE(t-t')+i\vec{p}(\vec{x}-\vec{x'})}}_{K(x-x')\,(t<t')} \gamma^0\, \underbrace{u(k) e^{-ik_0 t' + i\vec{k}\vec{x'}}}_{\phi(t',\vec{x'})}$$

$$= -i^2 \int \mathrm{d}^3\vec{p}\, \underbrace{(2\pi)^{-3} \int \mathrm{d}\vec{x'}\, e^{i(\vec{k}-\vec{p})\vec{x'}}}_{\delta^3(\vec{k}-\vec{p})} \frac{\left(-\gamma^0 E - \vec{\gamma}\vec{p} + m\right)\gamma^0 u(k)}{2E} \cdot e^{+iEt+i\vec{p}\vec{x}} \cdot e^{-i(k_0+E)t'}$$

$$= \frac{\left(-\gamma^0 k_0 - \vec{\gamma}\vec{k} + m\right)\gamma^0 u(k)}{2k_0} \cdot e^{-ik_0 t + i\vec{k}\vec{x}} \cdot e^{-2ik_0(t'-t)}$$

$$= \frac{\gamma^0\left(-\gamma^0 k_0 + \vec{\gamma}\vec{k} + m\right) u(k)}{2k_0} \cdot e^{-ik_0 t + i\vec{k}\vec{x}} \cdot e^{-2ik_0(t'-t)}$$

$$= 0$$

With

$$\left(\vec{\gamma}\vec{k} + m\right) u(k) = \gamma^0 k_0 u(k)$$

the term in braces in the fifth line always equals 0. By the choice of the integration paths we selected a solution of Equation (1) that corresponds to a field with positive energy that travels forward in time. In analogy the other solutions given in the lecture can be shown explicitly.
What we have shown by explicit calculation here can also be seen directly from what has been discussed in the lecture (slides 19ff): Note that for positive energy the pole sits at $p_0 = +E$. Following the integration path as outlined in the lecture for the evolution forward in time ($t > t'$) everything is clear: we close the contour in the lower half-plane (with $Im(f) \to -\infty$) and include the pole at $p_0 = +E$ in the contour. For the evolution backward in time we have to close the integration contour in the upper half-plane (with $Im(f) \to +\infty$). We would include a pole at $p_0 = -E$ if there were any. But we are in the case of positive energy. There is no pole within the integration contour and the integration result is always 0, irrespective of how we integrate in this case (as long as we remain above the real axis for $p_0 > 0$).