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Relativistic Quantum Mechanics

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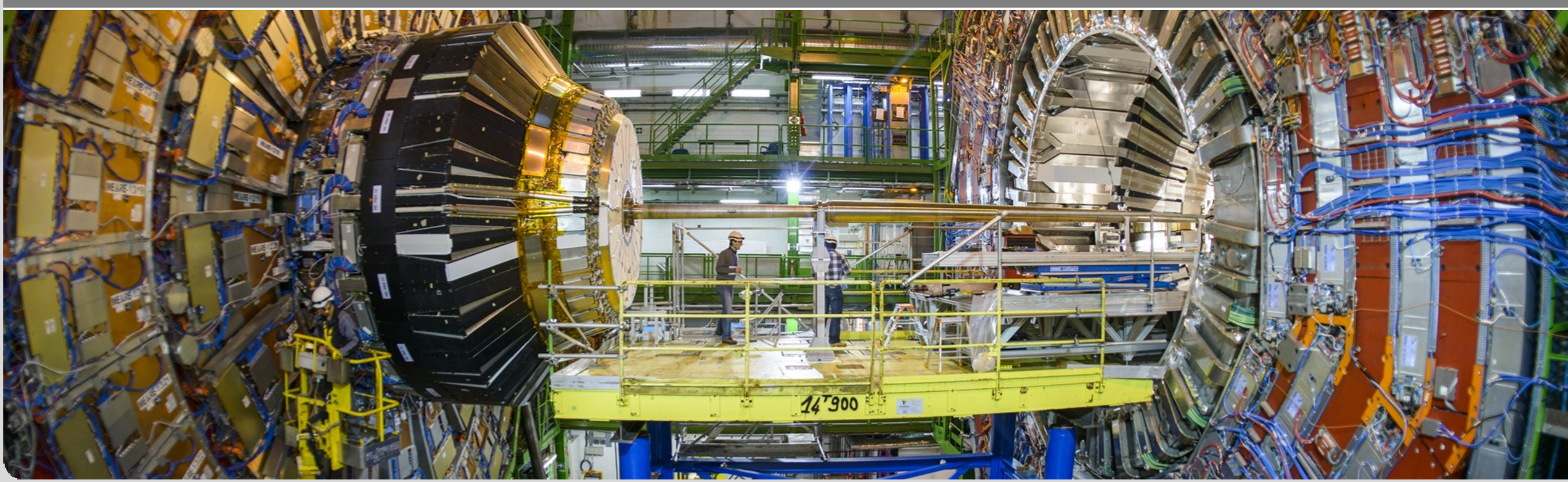
21. April 2016

INSTITUTE OF EXPERIMENTAL PARTICLE PHYSICS (IEKP) – PHYSICS FACULTY



Schedule for today

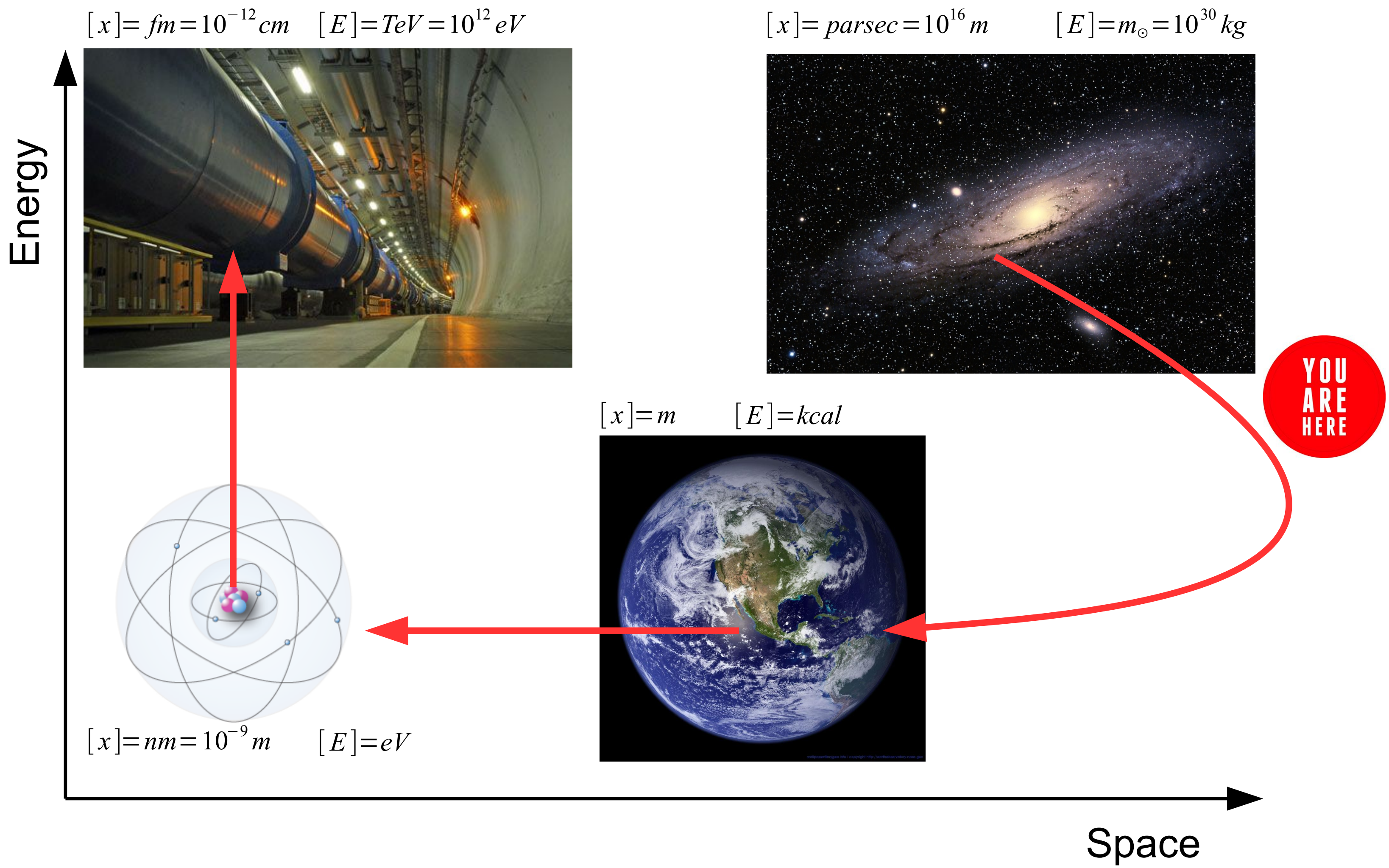
- What is the smallest dimension of the Dirac equation?
- What is the defining characteristic of a spinor?

3 Bosons & Fermions

2 Dirac EQ

1 Klein-Gordon EQ

Between cosmos & particle physics



Relativistic quantum mechanics

$[x] = fm = 10^{-12} cm \quad [E] = TeV = 10^{12} eV$

Natural units ($\rightarrow \hbar = 1, c = 1$):

$[m] = \mathrm{GeV} \qquad [x] = ?$

$[E] = \mathrm{GeV} \qquad [t] = ?$

$[p] = \mathrm{GeV} \qquad [\partial_\mu] = ?$

$\Delta p \cdot \Delta x \gtrsim \hbar$ ($\rightarrow$ *uncertainty relation*)

Smallest scales (10^{-12}cm).	+	Largest energies (10^{+12}eV).

($\rightarrow$ *Quantum Mechanics*)

($\rightarrow$ *relativistic dispersion relation* $E^2 = p^2 + m^2$)

- Most important Eq's to describe particle dynamics: *Klein-Gordon*, *Dirac* Eq.

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The Klein-Gordon equation

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Canonical Operator Replacement

Application to Wave Function

$$E^2 - p^2 = m^2 \quad \Rightarrow \quad \boxed{\begin{aligned} E &\to i\partial_t \\ \vec{p} &\to -i\vec{\nabla} \end{aligned}} \quad \Rightarrow \quad \boxed{\left(\partial_\mu \partial^\mu + m^2\right)\phi = 0}$$

(*Klein-Gordon Eq*)

- Solutions:

$$\phi_+(\vec{x},t) = u(\vec{p})e^{+i(\vec{p}\vec{x}-Et)}$$

$$\phi_-(\vec{x},t) = v(\vec{p})e^{-i(\vec{p}\vec{x}-Et)} \qquad E(\vec{p}) = \sqrt{m^2+\vec{p}^2}$$

(*free wave*)

- Peculiarity: Hamiltonian is non-local.

$$\hat{H}_0 = \sqrt{m^2 - \vec{\nabla}^2} = m\sqrt{1 - \frac{\vec{\nabla}^2}{m^2}} = m - \frac{\vec{\nabla}^2}{2m} + \cdots \quad (*)$$

The Dirac equation

- Historical approach by *Paul Dirac 1927*: Find representation of relativistic dispersion relation, which is linear in space time derivatives:

$$\boxed{i\partial_t\psi = \hat{H}_0\psi = \left(-i\vec{\alpha}\vec{\nabla} + \beta m\right)\psi}$$

- Cannot be pure numbers (*) (→ *algebraic operators*).
- Need four independent operators.

- Require Klein-Gordon Eq to be fulfilled for a free Dirac particle:

$$(i\partial_t)^2\psi = \left(-i\vec{\alpha}\vec{\nabla} + \beta m\right)^2\psi$$

$$= \left[-\underbrace{(\alpha_i\alpha_j + \alpha_j\alpha_i)}_{}\underset{(i\leq j)}{\partial_i\partial_j} - im\underbrace{(\alpha_i\beta + \beta\alpha_i)}_{}\partial_i + \underbrace{(\beta m)^2}_{}\right]\psi \overset{!}{\equiv} \left[-\vec{\nabla}^2 + m\right]\psi$$

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \qquad \{\alpha_i, \beta\} = 0 \qquad \beta^2 = 1$$

(Anti-commutation relations)

Properties of $\vec{\alpha}$ and β

- Operators $\vec{\alpha}$ and β can be expressed by matrices:

  Must be hermitian: $\hat{H}_0$ should have real *eigenvalues*.

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  Must be traceless:

$$\boxed{Tr(\alpha_i) = Tr(\alpha_i \overbrace{\beta\beta}^{\mathbb{I}}) \underset{\substack{\textit{cyclic}\\ \textit{permutation}}}{=} Tr(\beta\alpha_i\beta) \underset{\substack{\textit{anti-commutator}\\ \textit{relation}}}{=} -Tr(\beta\beta\alpha_i) = -Tr(\alpha_i) = 0}$$

Properties of $\vec{\alpha}$ and β

- Operators $\vec{\alpha}$ and β can be expressed by matrices:

  Must have at least dim=4:

  - $\alpha_i^2 = \mathbb{I} \rightarrow$ has only eigenvectors ±1.
  - $\beta^2 = \mathbb{I} \rightarrow$ has only eigenvectors ±1.
  - Dimension must be even to obtain 0 trace.
  - $\mathbb{I}$ + Pauli matrices $(\mathbb{I}, \sigma_i)$ form a basis of the space of 2×2 matrices. But $\mathbb{I}$ is not traceless ($\rightarrow$ no chance to obtain four independent(!) traceless matrices).
  - Simplest representation must at least have dim=4 (can be higher dimensional though).

The Dirac representation

- Concrete representation of α_i and β matrices:

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

($\sigma_i (i = 1, 2, 3)$ *Pauli matrices*)

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Dirac representation

- Relativistic formulations use γ^{μ} [(1)] matrices:

$$\gamma^0 \equiv \beta \qquad \gamma^i \equiv \beta\alpha_i$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$$\boxed{\begin{array}{l} \{\alpha_i, \alpha_j\} = 2\delta_{ij} \\ \{\alpha_i, \beta\} = 0 \\ [\,\beta\,,\,\beta\,] = 0 \end{array} \quad \Rightarrow \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}}$$

(*Compact notation of algebra*)

- Dirac equation in mostly known covariant form:

$$\boxed{(i\gamma^\mu \partial_\mu - m)\,\psi = 0}$$

(*Dirac Eq*)

(1) Formally transformation behavior of a Lorentz-vector.

Solutions of the Dirac equation

$$\boxed{\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0}$$

$$\psi_+(\vec{x}) = u(\vec{p})e^{+i(\vec{p}\vec{x}-Et)}$$

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(*free wave*)

$$\Big\downarrow \text{Spinors}$$

$$\boxed{\begin{array}{lll} \text{for } +m & u_\uparrow(0) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} & u_\downarrow(0) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \\ \text{for } -m & v_\uparrow(0) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} & v_\downarrow(0) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \end{array}}$$

at rest ($\vec{p} \equiv 0$)

Solutions of the Dirac equation

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(*free wave*)

Spinors

(*Lorentz Transformation*)

$\Lambda : (m,0,0,0) \rightarrow (E, p_x, p_y, p_z)$

for $+m$

$$u_\uparrow(0) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad u_\downarrow(0) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

for $-m$

$$v_\uparrow(0) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad v_\downarrow(0) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

at rest $(\vec{p} \equiv 0)$

for $+m$

$$u_\uparrow(\vec{p}) = N\begin{pmatrix}E+m\\0\\p_z\\p_x+ip_y\end{pmatrix} \qquad u_\downarrow(\vec{p}) = N\begin{pmatrix}0\\E+m\\p_x-ip_y\\-p_z\end{pmatrix}$$

for $-m$

$$v_\uparrow(\vec{p}) = N\begin{pmatrix}p_z\\p_x+ip_y\\E+m\\0\end{pmatrix} \qquad v_\downarrow(\vec{p}) = N\begin{pmatrix}p_x-ip_y\\-p_z\\0\\E+m\end{pmatrix}$$

$$N = \frac{1}{\sqrt{2m(E+m)}}$$

in motion $(\vec{p} \neq 0)$

The transformation behavior of spinors

$$\boxed{\begin{array}{ll} \Lambda: \; x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu & \text{Lorentz vector} \\ \psi_\alpha(x) \rightarrow \psi'_\alpha(x') = S_{\alpha\beta}(\Lambda)\psi_\beta(\Lambda x) & \text{Spinor} \end{array}}$$

(*Lorentz transformation*)

mixes components of ψ

acts on coordinates

- How does $S(\Lambda)$ look like?

$$S(\Lambda) = e^{-\frac{i}{4} r_{\mu\nu}\sigma^{\mu\nu}} = \begin{cases} e^{-i\vec{\varphi}\cdot\left(\frac{1}{2}\vec{\Sigma}\right)} \\ e^{\hat{v}\cdot\frac{1}{2}\vec{\alpha}} \end{cases}$$

Spacial rotation

$$\cos\left(\frac{\varphi}{2}\right) - i\sin\left(\frac{\varphi}{2}\right)\left(\hat{\varphi}\cdot\vec{\Sigma}\right)$$

Rotation of 2π around spacial quantization axis turns $\psi_\alpha(x) \rightarrow -\psi_\alpha(x)$.

Boost

$$\cosh\left(\frac{v}{2}\right) + \sinh\left(\frac{v}{2}\right)\left(\hat{v}\cdot\vec{\alpha}\right)$$

Classification of physical objects

- In physics we classify objects according to their transformation behavior.

- (*Lorentz-*)Scalar:
- (*Lorentz-*)Vector:
- (*Lorentz-*)Tensor (2. order):
- (*Lorentz-*)Spinor: $\Lambda: \quad \psi^{\alpha}(x^{\mu}) \rightarrow \psi'^{\alpha}(x'^{\mu}) = S^{\alpha}_{\beta}\psi^{\beta}(\Lambda^{\mu}_{\nu}x^{\nu})$

Classification of physical objects

- In physics we classify objects according to their transformation behavior.

- (*Lorentz*-)Scalar: $\Lambda: \quad m \quad \rightarrow m' \quad = m$
- (*Lorentz*-)Vector: $\Lambda: \quad x^\mu \quad \rightarrow x'^\mu \quad = \Lambda^\mu_\nu x^\nu$
- (*Lorentz*-)Tensor (2. order): $\Lambda: \quad F^{\mu\nu} \quad \rightarrow F'^{\mu\nu} \quad = \Lambda^\mu_\alpha \Lambda^\nu_\beta F^{\alpha\beta}$
- (*Lorentz*-)Spinor: $\Lambda: \quad \psi^\alpha(x^\mu) \rightarrow \psi'^\alpha(x'^\mu) = S^\alpha_\beta \psi^\beta(\Lambda^\mu_\nu x^\nu)$

Objects composed from spinors

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- Observables can be composed of *spinors*:

$\overline{\psi} = \psi^\dagger \gamma^0$ (*Adjoint spinor*)

$\overline{\psi}\psi$	?
$\overline{\psi}\gamma^5\psi$	?
$\overline{\psi}\gamma^\mu\psi$	?
$\overline{\psi}\gamma^5\gamma^\mu\psi$	?
$\overline{\psi}\sigma^{\mu\nu}\psi$	?

Objects composed from spinors

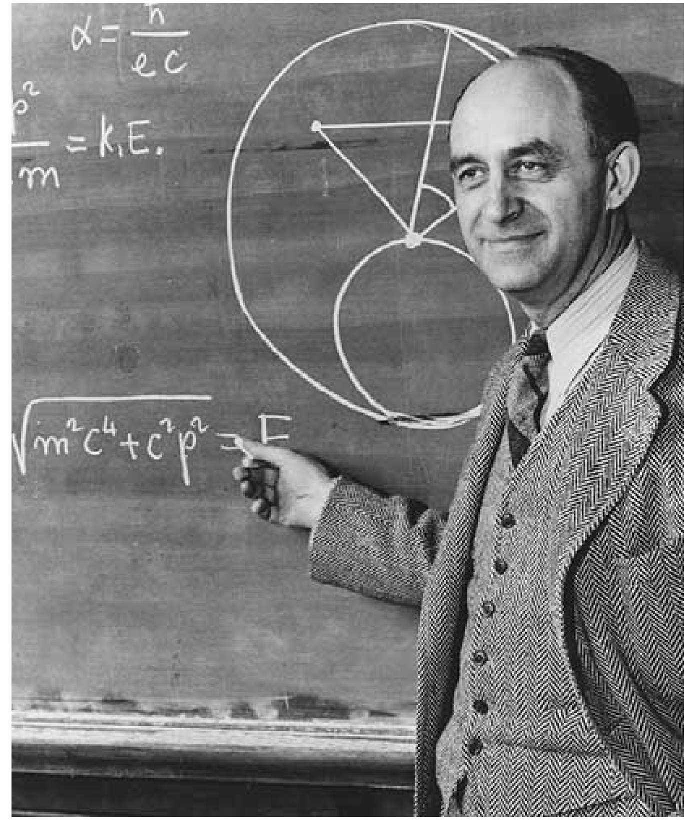
- Observables can be composed of *spinors*:

$\overline{\psi} = \psi^\dagger \gamma^0$ (*Adjoint spinor*)

$\overline{\psi}\psi$	Scalar
$\overline{\psi}\gamma^5\psi$	Pseudo Scalar
$\overline{\psi}\gamma^\mu\psi$	Vector
$\overline{\psi}\gamma^5\gamma^\mu\psi$	Axial Vector
$\overline{\psi}\sigma^{\mu\nu}\psi$	Tensor (2. order)

Bosons & Fermions

Satyenda Nath Bose
(*1. January 1894, † 4. February 1974)



Enrico Fermi
(*29. September 1901, † 28. November 1954)

Bosons	Fermions
$$\boxed{\left(\partial_\mu \partial^\mu + m^2\right) \phi = 0}$$	$$\boxed{\left(i\gamma^\mu \partial_\mu - m\right) \psi = 0}$$
• Integer spin 0, 1, ...[(2)]	• Half-integer spin ½, ...[(2)]
• Commutator relations [. , .].	• Anti-commutator relations { . , . }.

[(2)] This holds for elementary particle as well as for pseudo-particles.

Bosons

$$\boxed{\left(\partial_\mu \partial^\mu + m^2\right)\phi = 0}$$

- Integer spin 0, 1, ...[(2)]
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Fermions

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- Anti-commutator relations { . , . }.

Multi-particle systems

Bosons

- Symmetric wave functions.
- Bose-Einsten statistics.
- More than one particle can be described by single wave function.

Fermions

- Anti-symmetric wave functions.
- Fermi statistics.
- Each particle occupies unique place in phasespace (→ *Pauli Principle*).

(2) This holds for elementary particle as well as for pseudo-particles.

Concluding remarks

- Brief reprise of main relativistic equations of quantum mechanics.
- Klein-Gordon equation to describe kinematics of bosons.
- Dirac equation to describe kinematics of fermions.
- Main tool box to go from field theoretical calculations to observable behavior of particles.
- From next lecture on go one level up → field theoretical level.
- Prepare "*The Higgs Boson Discovery at the Large Hadron Collider*" Section 2.1.

Backup

Dirac representation

- The following 16 matrices form a basis of the 4×4 matrices:

$\mathbb{I}_4$	1 matrix
γ^μ	4 matrices
$\sigma^{\mu\nu} \equiv \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$	6 matrices
$\gamma^\mu \gamma^5$	4 matrices
$\gamma^5 \equiv \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$	1 matrix

- Orthonormal (with scalar product $< .|. >= \frac{1}{4}Tr(. \cdot .)$).
- All matrices traceless apart from $\mathbb{I}_4$.