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# Lagrange Formalism & Gauge Theories

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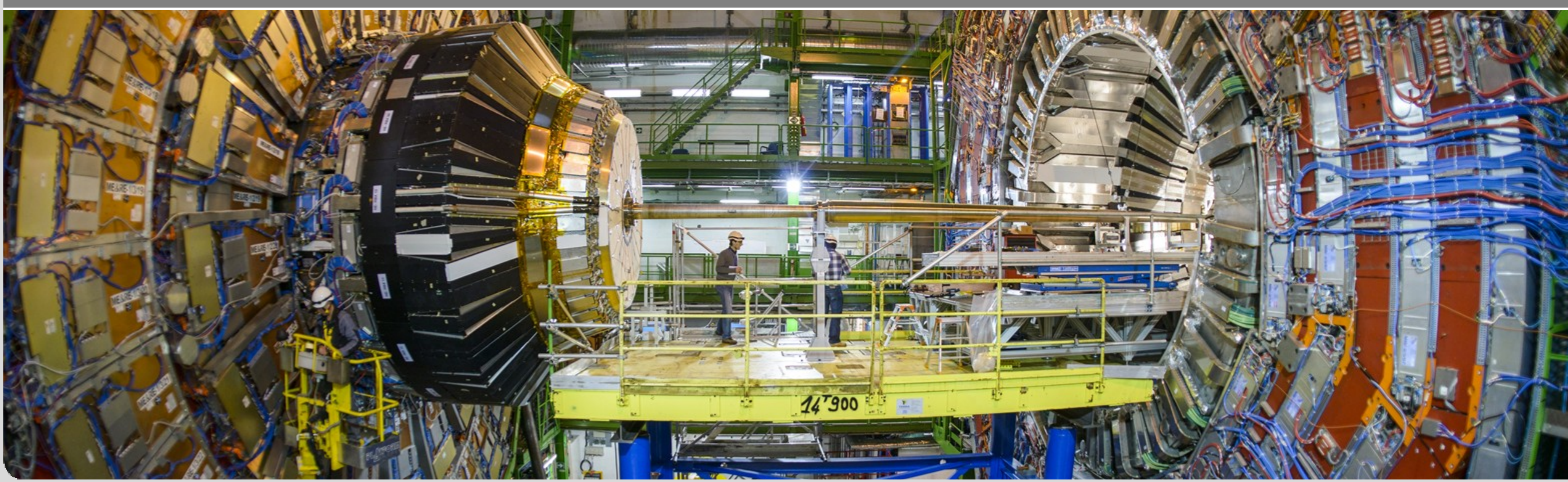
28. April 2015

INSTITUTE OF EXPERIMENTAL PARTICLE PHYSICS (IEKP) – PHYSICS FACULTY



# Schedule for today

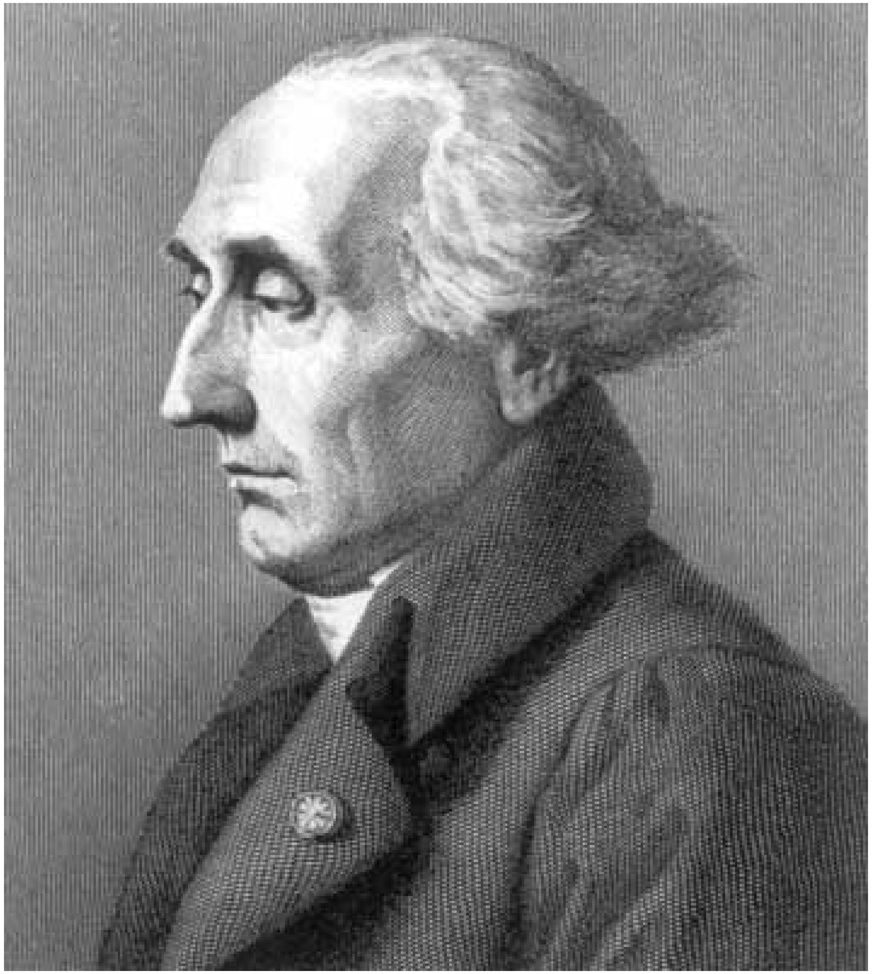
- How do I know that the gauge field should be a boson?
- What is the defining characteristic of  a Lie group?

3 Lie-groups & (Non-)Abelian transformations

2 Local gauge transformations

1 Lagrange formalism

# Lagrange formalism & gauge transformations



Joseph-Louis Lagrange
(*25. January 1736, † 10. April 1813)

# Lagrange formalism (classical field theories)

- All information of a physical system is contained in the *action* integral:

$$\mathcal{S} = \int \mathcal{L}(\partial_\mu \phi, \phi) \mathrm{d}x_\mu$$

*Action:* $\mathcal{S}$

*Field:* $\phi$ (*Generalization of canonical coordinates*)

*Lagrange Density:* $\mathcal{L}$
( $\sum E_{\text{kin}} - \sum E_{\text{pot}}$ )

- Equations of motion can be derived from the *Euler-Lagrange formalism*:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

*(From variation of action)*

- What is the dimension of $\mathcal{L}$?

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*(From variation of action)*

- What is the dimension of $\mathcal{L}$? → $[\mathcal{L}] = \mathrm{GeV}^4$

# Lagrange density for (free) bosons & fermions

For Bosons:

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^*$$

For Fermions:

$$\mathcal{L} = \overline{\psi} \left( i\gamma^\mu \partial_\mu - m \right) \psi$$

- Proof by applying *Euler-Lagrange formalism* (shown only for Bosons here):

$$\partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^*)} - \frac{\partial \mathcal{L}}{\partial \phi^*} = 0$$

$$\partial^\mu \partial_\mu \phi \qquad -m^2 \phi \longrightarrow \left( \partial^\mu \partial_\mu + m^2 \right) \phi = 0$$

- **NB:**
  - There is a distinction between $\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$ and $\partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^*)}$ .
  - Most trivial is variation by $\overline{\psi}$ , least trivial is variation by $\psi$.

# Global phase transformations

- The Lagrangian density is covariant under global phase transformations (shown here for the fermion case only):

$$\psi(\vec{x},t) \rightarrow \psi'(\vec{x},t) = e^{i\vartheta}\psi(\vec{x},t)$$

$$\overline{\psi}(\vec{x},t) \rightarrow \psi'(\vec{x},t) = \overline{\psi}(\vec{x},t)e^{-i\vartheta}$$

(*Global phase transformation*)

$$\vartheta \neq \vartheta(\vec{x},t)$$

$$\mathcal{L}' = \overline{\psi'}\left(i\gamma^\mu\partial_\mu - m\right)\psi' = \overline{\psi}e^{-i\vartheta}\left(i\gamma^\mu\partial_\mu - m\right)e^{i\vartheta}\psi$$

$$= \overline{\psi}\left(i\gamma^\mu\ \partial_\mu \qquad\qquad - m\right)\psi \ \ = \mathcal{L}$$

- Here the phase $\vartheta$ is fixed at each point in space $\vec{x}$ at any time $t$ .
- What happens if we allow different phases at each point in $(\vec{x},t)$?

# Local phase transformations

- This is not true for local phase transformations:

$$\psi(\vec{x},t) \rightarrow \psi'(\vec{x},t) = e^{i\vartheta}\psi(\vec{x},t)$$

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(*Local phase transformation*)

$$\vartheta = \vartheta(\vec{x},t)$$

$$\mathcal{L}' = \overline{\psi'}\,(i\gamma^\mu\partial_\mu - m)\,\psi' = \overline{\psi}e^{-i\vartheta}\,(i\gamma^\mu\partial_\mu - m)\,e^{i\vartheta}\psi$$

$$= \overline{\psi}\,(i\gamma^\mu(\partial_\mu + i\partial_\mu\vartheta) - m)\,\psi \quad \neq \mathcal{L}$$

*Breaks invariance due to* $\partial_\mu$ *in* $\mathcal{L}$. → $\frac{\psi(x+\Delta x)-\psi(x)}{\Delta x}$

*Connects neighboring points in* $(\vec{x},t)$

# The covariant derivative

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- Covariance can be enforced by the introduction of the covariant derivative: $\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$ with the corresponding transformation behavior

$$\psi(\vec{x},t) \to \psi'(\vec{x},t) = e^{i\vartheta}\psi(\vec{x},t)$$

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$$D_\mu \to D'_\mu = D_\mu - i\partial_\mu\vartheta$$

(*Local phase transformation*)

$$\vartheta = \vartheta(\vec{x},t)$$

(Arbitrary *gauge field*)

$$\mathcal{L}' = \overline{\psi'}\left(i\gamma^\mu D'_\mu - m\right)\psi' = \overline{\psi}e^{-i\vartheta}\left(i\gamma^\mu (D_\mu - i\partial_\mu\vartheta) - m\right)e^{i\vartheta}\psi$$

$$= \overline{\psi}\left(i\gamma^\mu (D_\mu - i\partial_\mu\vartheta + i\partial_\mu\vartheta) - m\right)\psi \quad = \mathcal{L}$$

- **NB:** What is the transformation behavior of the gauge field $A_\mu$?

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$$= \overline{\psi}\left(i\gamma^\mu(D_\mu - i\partial_\mu\vartheta + i\partial_\mu\vartheta) - m\right)\psi \quad = \mathcal{L}$$

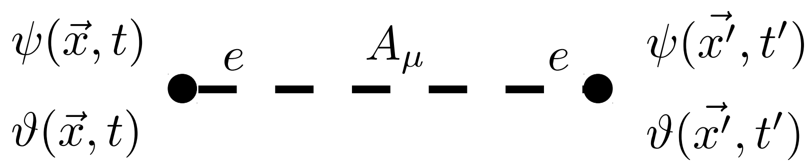
- **NB:** What is the transformation behavior of the gauge field $A_\mu$?

$A_\mu \to A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\vartheta$ ⟶ known from electro-dynamics!

# The gauge field



- Possible to allow arbitrary phase $\vartheta$ of $\psi(\vec{x},t)$ at each individual point in $(\vec{x},t)$ .
- Requires introduction of a mediating field $A_\mu$, which transports this information from point to point.

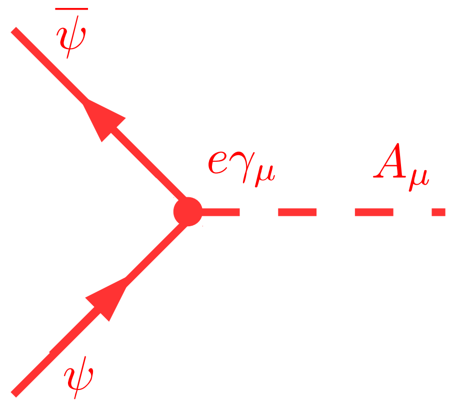



- The gauge field $A_\mu$ couples to a quantity $e$ of the external field $\psi(\vec{x},t)$, which can be identified as the electric charge.
- The gauge field $A_\mu$ can be identified with the photon field.

# The interacting fermion

- The introduction of the covariant derivative leads to the *Lagrangian density* of an interacting fermion with electric charge $e$:

$$\mathcal{L}_{\mathrm{IA}} = \overline{\psi}\,(i\gamma^{\mu}(D_{\mu} - m)\,\psi$$
$$= \underbrace{\overline{\psi}\,(i\gamma^{\mu}\ \partial_{\mu} - m)\,\psi}_{\text{free fermion field}} - \underbrace{e\overline{\psi}\gamma^{\mu}A_{\mu}\psi}_{\text{IA term}}$$




- Description still misses dynamic term for a free *gauge boson field* (=photon).

# Gauge field dynamics

- Ansatz:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad\qquad \mathcal{L}_{\text{kin}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

(*Field-Strength tensor*) (*Free photon field*)

- Motivation:

- Variation of the action integral
  $\mathcal{S} = \delta \int (-m\text{d}s - eA_\mu \text{d}x^\mu)$
  in classical field theory, leads to
  $m\frac{dv_\mu}{ds} = e(\partial_\mu A_\nu - \partial_\nu A_\mu)v^\nu$
- Can also be obtained from:
  $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) = \frac{i}{e}[D_\mu, D_\nu]$

- $F_{\mu\nu}F^{\mu\nu}$ is manifest Lorentz invariant.
- $A_\mu$ appears quadratically → linear appearance in variation that leads to equations of motion (→ superposition of fields).
- Check that $F_{\mu\nu}$ is gauge invariant.

# Complete Lagrangian density

- Application of $U(1)$ gauge symmetry leads to Largangian density of QED:

$$\mathcal{L}_{\text{QED}} = \overline{\psi}\,(i\gamma^{\mu}(D_{\mu} - m)\,\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$$
$$= \underbrace{\overline{\psi}\,(i\gamma^{\mu}\quad\partial_{\mu} - m)\,\psi}_{\text{Free Fermion Field}} - \underbrace{e\overline{\psi}\gamma^{\mu}A_{\mu}\psi}_{\text{IA Term}} - \underbrace{\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{Gauge}}$$

*(Interacting Fermion)*

- Variation of $\overline{\psi}$ :

$$i\gamma^{\mu}\,(\partial_{\mu} - m)\,\psi + e\gamma^{\mu}A_{\mu}\psi = 0$$

- Derive equations of motion for an interacting boson.

# Complete Lagrangian density

- Application of $U(1)$ gauge symmetry leads to Largangian density of QED:

$$\mathcal{L}_{\text{QED}} = \overline{\psi}\,(i\gamma^\mu(D_\mu - m)\,\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$
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*(Interacting Fermion)*

- Variation of $A_\mu$:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \partial_\mu F^{\mu\nu} = 0$$

$$\partial_\mu\,(\partial^\mu A^\nu - \partial^\nu A^\mu) = (\partial_\mu \partial^\mu A_\mu - \partial^\nu \underbrace{\partial_\mu A^\mu}) = 0$$

$\partial_\mu A^\mu = 0$ *(Lorentz Gauge)*

$$(\partial_\mu \partial^\mu - 0)\,A_\mu = 0$$

*(Klein-Gordon equation for a massless particle)*

# Summary (Abelian) gauge field theories

$$\psi(\vec{x},t) \rightarrow \psi'(\vec{x},t) = e^{i\vartheta}\psi(\vec{x},t)$$
$$\overline{\psi}(\vec{x},t) \rightarrow \psi'(\vec{x},t) = \overline{\psi}(\vec{x},t)e^{-i\vartheta}$$

→ (*Local gauge invariance*)

$$\partial_\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu$$
$$D_\mu \rightarrow D'_\mu = D_\mu - i\partial_\mu\vartheta$$
$$A_\mu \rightarrow A'_\mu = A_\mu - \tfrac{1}{e}\partial_\mu\vartheta$$

→ (*Covariant derivative*)

$$F_{\mu\nu} \equiv [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu$$
$$F_{\mu\nu} \rightarrow F'_{\mu\nu} = F_{\mu\nu}$$

→ (*Field strength tensor*)

$$\mathcal{L} = \overline{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

→ (*Lagrange density*)

# Review of Lie-Groups

Marius Sophus Lie
(*17. December 1842, † 18. February 1899)

# Unitary transformations

$U(1)$ phase transformation.

$$\psi(\vec{x},t) \rightarrow \psi'(\vec{x},t) = e^{i\vartheta}\psi(\vec{x},t)$$

- $U(1)$ is a group of unitary transformations in $\mathbb{R}^n$ with the following properties:

$$\mathbf{G} \in U(n) \qquad \mathbf{G}^\dagger \mathbf{G} = \mathbb{I}_n \qquad |\det \mathbf{G}| = 1$$

- Splitting an additional phase from $\mathbf{G}$ one can reach that $\det\mathbf{G} = 1$:

$$U(n) = U(1) \times SU(n)$$

$|\det \mathbf{G}| = 1$ *(Unitary transformations)*

$\det \mathbf{G} = +1$ *(Special unitary transformations)*

# Infinitesimal → finite transformations

- The $SU(n)$ can be composed from infinitesimal transformations with a continuous parameter $\vartheta \in \mathbb{R}$ :

$$\mathbf{G}\big|_{\text{finite}} = \mathbb{I}_n + i\vartheta_{\text{finite}}\mathbf{t} \qquad (\vartheta_{\text{finite}} \in \mathbb{R}\,,\ \mathbf{t} \in \mathcal{M}(\mathrm{n}\times\mathrm{n}))$$

$$\mathbf{G}\big|_{\text{finite}} = \left(\mathbb{I}_n + i\frac{\vartheta_{\text{finite}}}{m}\mathbf{t}\right)^m \xrightarrow{m\to\infty} e^{i\vartheta_{\text{finite}}\cdot\mathbf{t}}$$

→ $\mathbf{t}$ generators of $\mathbf{G}$.

→ $\mathbf{t}$ define structure of $\mathbf{G}$.

- The set of $\mathbf{G}$ forms a *Lie-Group*.
- The set of $\mathbf{t}$ forms the tangential-space or *Lie-Algebra*.

# Properties of $\mathrm{t}$

- Hermitian:

$$\mathbf{G}^\dagger\mathbf{G} = \mathbb{I}_n$$
$$= \left(\mathbb{I}_n - i\vartheta\mathbf{t}^\dagger\right)\left(\mathbb{I}_n + i\vartheta\mathbf{t}\right) = \mathbb{I}_n + i\vartheta\underbrace{\left(\mathbf{t}-\mathbf{t}^\dagger\right)}_{} + O(\vartheta^2)$$

$$\boxed{\mathbf{t} = \mathbf{t}^\dagger}$$

- Traceless (example $SU(n)$):

$$\det\mathbf{G} = \det\left(\mathbb{I}_n + i\vartheta\mathbf{t}\right)$$
$$= 1 + i\vartheta\mathrm{Tr}(\mathbf{t}) + O(\vartheta^2) \overset{!}{=} 1$$

$$\boxed{Tr(\mathbf{t}) = 0}$$

- Dimension of tangential space:

$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & \\ * & * & * & & \\ * & * & & & \\ * & & & & \end{pmatrix}$$

- $n$ real entries in diagonal.
- $1/2 \cdot n(n-1)$ complex entries in off-diagonal.
- $-1$ for $SU(n)$ for det req.

> - $U(n)$ has $n^2$ generators.
> - $SU(n)$ has $(n^2-1)$ generators.

# Examples that appear in the SM ( $U(1)$ )

- $U(1)$ transformations (equivalent to $O(2)$ ):
  - Number of generators: $1^2 \quad = 1$ **NB:** what is the Generator?

# Examples that appear in the SM ( $U(1)$ )

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- $U(1)$ transformations (equivalent to $O(2)$ ):
  - Number of generators: $1^2 \qquad = 1$ **NB:** what is the Generator? ⟶ The generator is 1.

# Examples that appear in the SM ($SU(2)$)

- $SU(2)$ transformations (equivalent to $O(3)$):
  - Number of generators: $(2^2 - 1) = 3$
  - i.e. there are 3 matrices $\{\mathbf{t}_j\}$ , which form a basis of traceless hermitian matrices, for which the following relation holds:

$$\mathbf{G} = e^{i\sum_{j=1}^{3}\vartheta_j \mathbf{t}_j}$$

  - Explicit representation:

$$\mathbf{t}_j = \frac{1}{2}\sigma_j \qquad (j = 1\ldots 3)$$

(*3 Pauli matrices*)

$$[\mathbf{t}_i, \mathbf{t}_j] = i\epsilon_{ijk}\mathbf{t}_k$$

- algebra closes.
- structure constants of $SU(2)$.

# Examples that appear in the SM ($SU(3)$)

- $SU(3)$ transformations (equivalent to $O(4)$):
  - Number of generators: $(3^2-1)=8$
  - i.e. there are 8 matrices $\{\mathbf{T}_j\}$, which form a basis of traceless hermitian matrices, for which the following relation holds:

$$\mathbf{G} = e^{i\sum_{j=1}^{8}\vartheta_j\mathbf{T}_j}$$

  - Explicit representation:

$$\mathbf{T}_j = \frac{1}{2}\lambda_j \quad (j = 1\ldots 8)$$

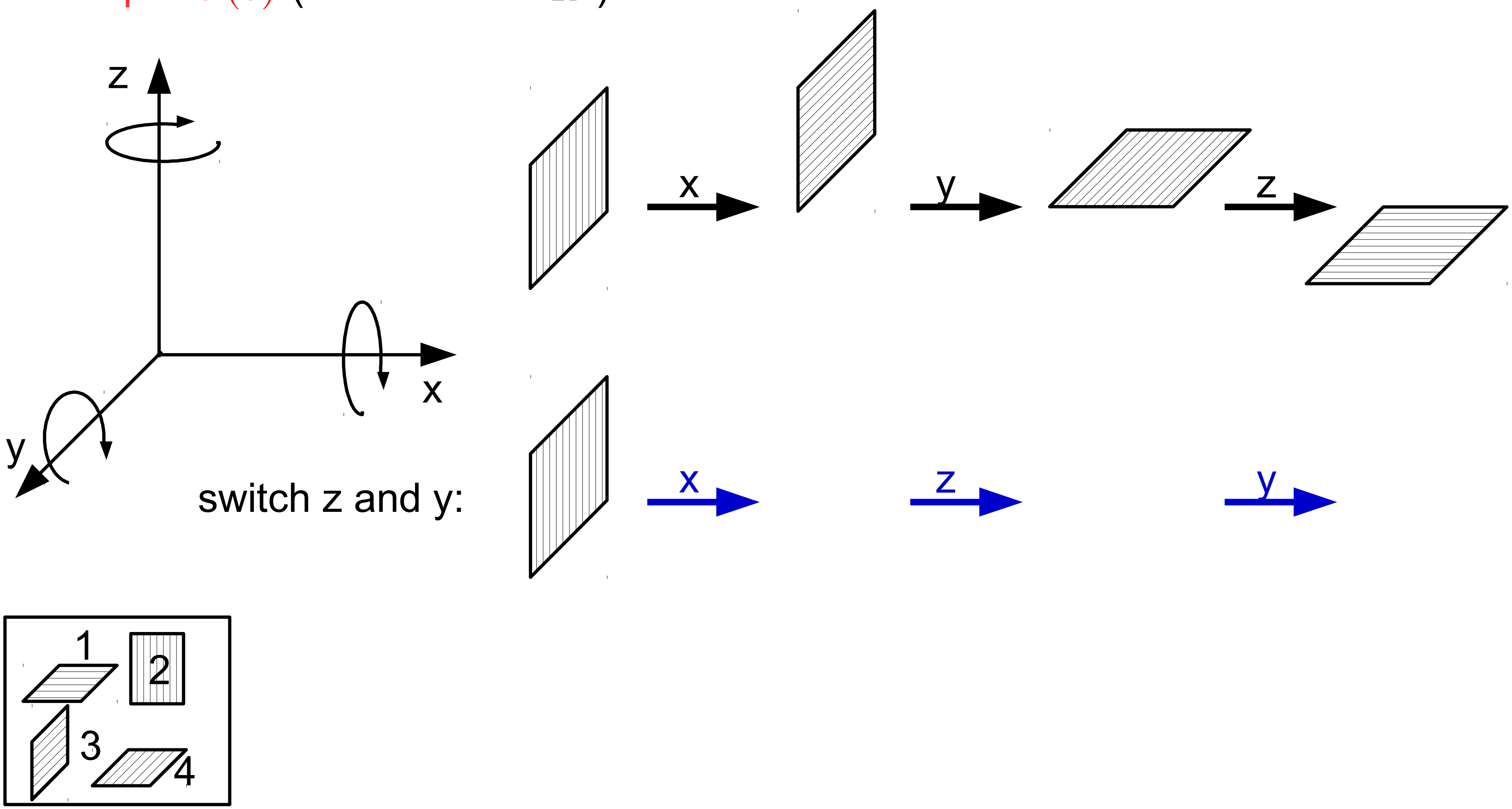
(*8 Gell-Mann matrices*)

$$[\mathbf{T}_i, \mathbf{T}_j] = i f_{ijk}\mathbf{T}_k$$

- algebra closes.
- structure constants of $SU(3)$.

# (Non-)Abelian symmetry transformations

- Example $O(3)$ (90° rotations in $\mathbb{R}^3$):



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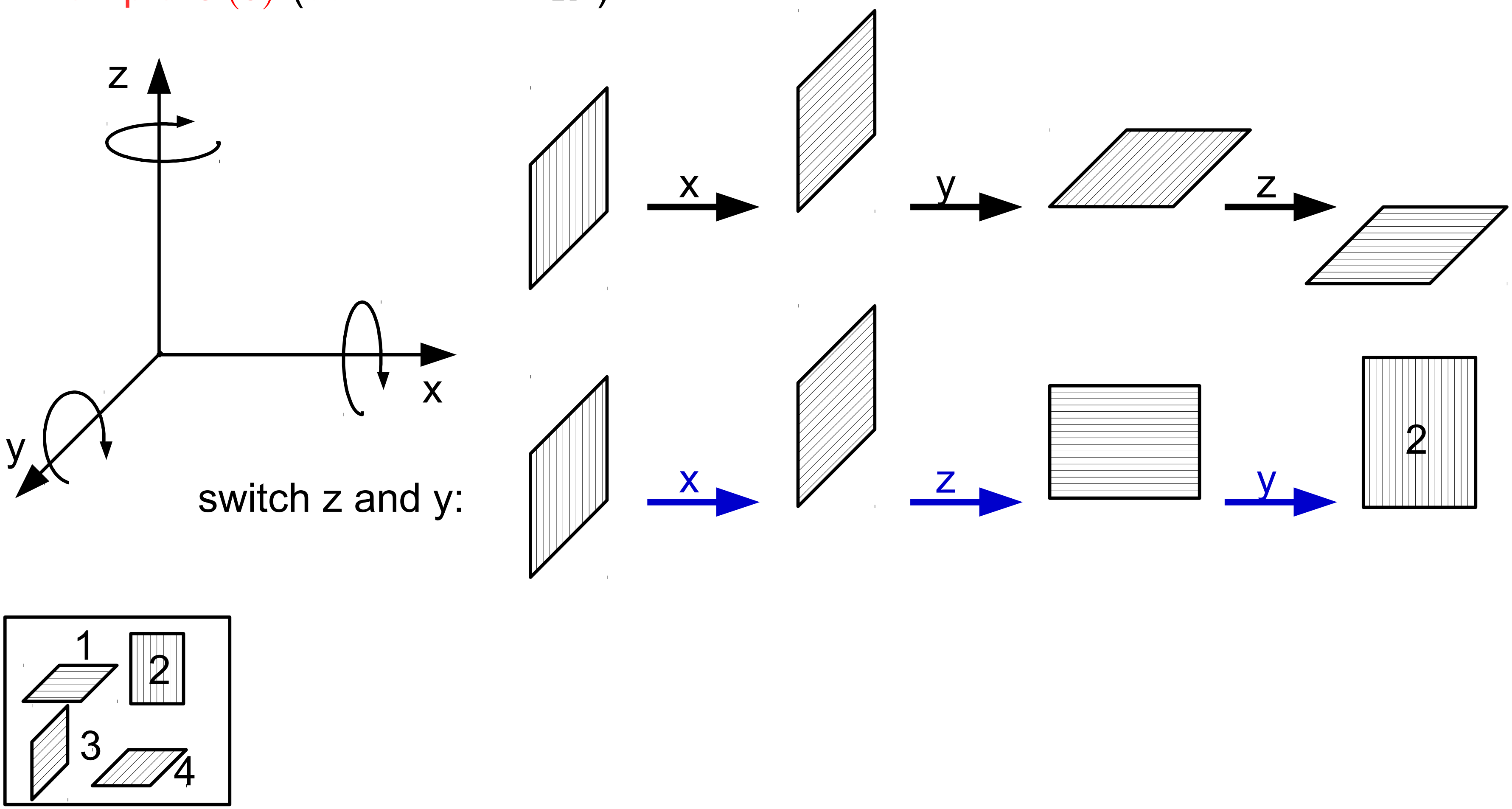

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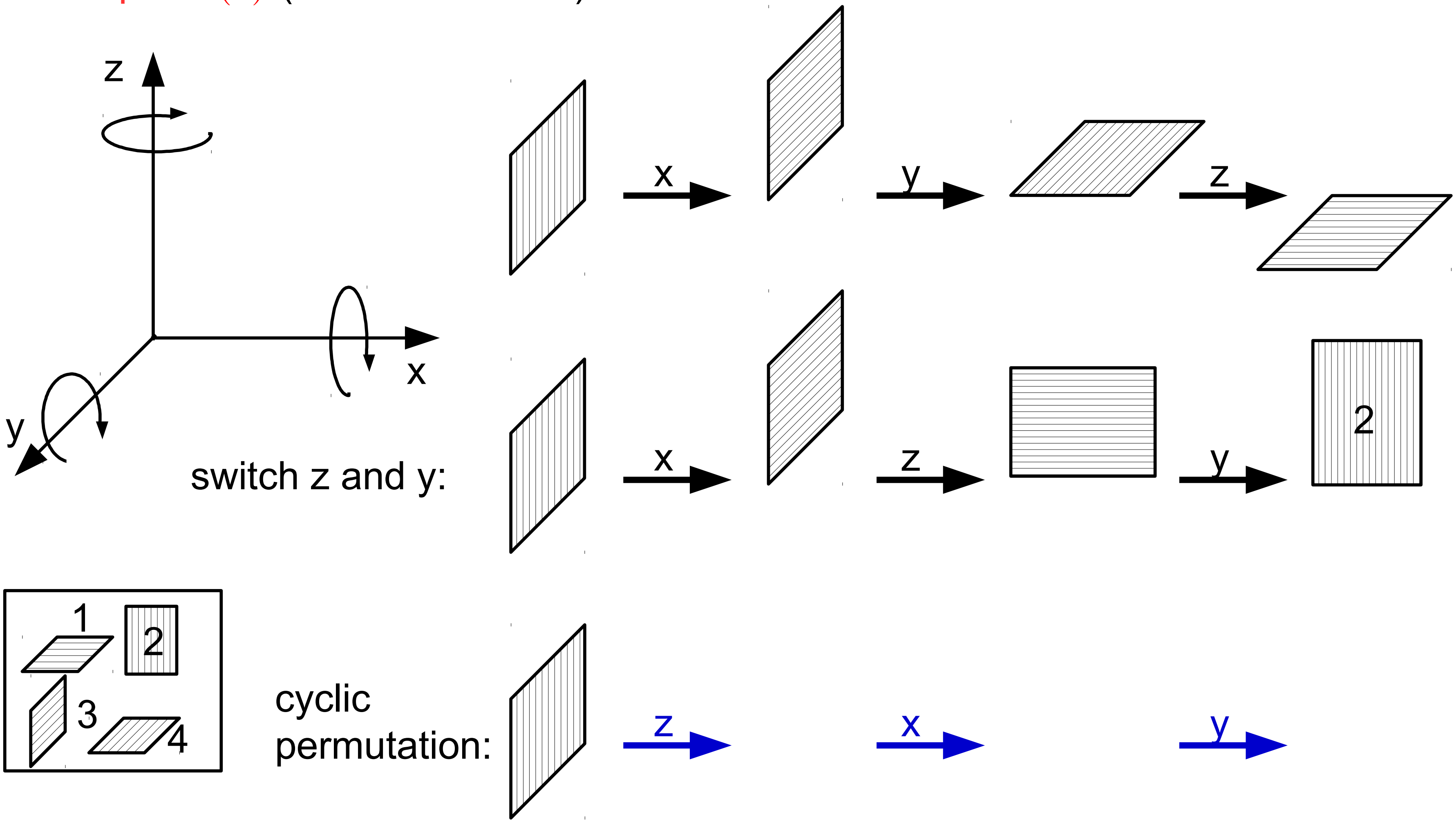

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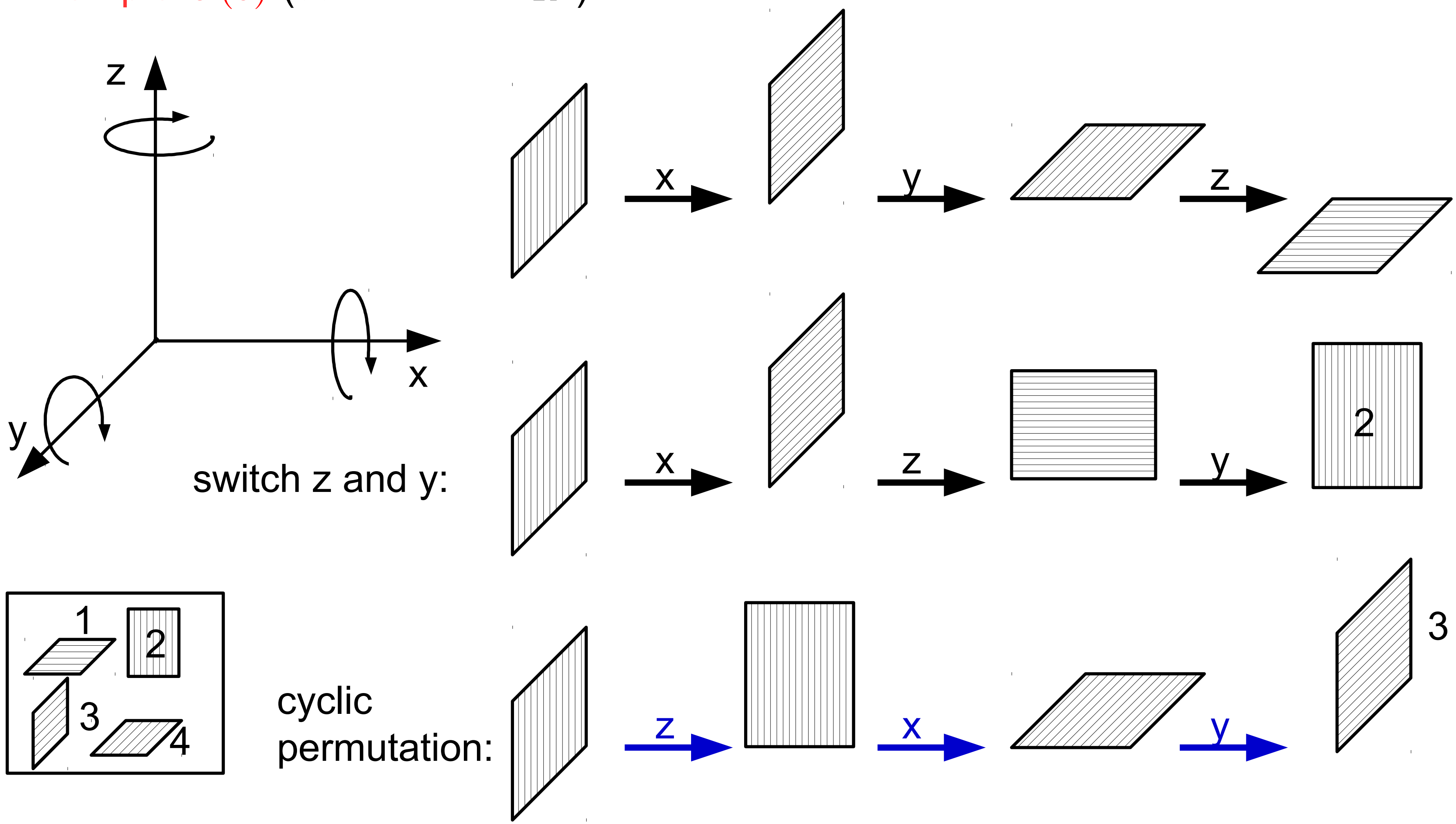

# Abelian vs. Non-Abelian gauge theories

$$\psi(\vec{x},t) \to \psi'(\vec{x},t) = e^{i\vartheta}\psi(\vec{x},t)$$
$$\overline{\psi}(\vec{x},t) \to \psi'(\vec{x},t) = \overline{\psi}(\vec{x},t)e^{-i\vartheta}$$

$$\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$$
$$D_\mu \to D'_\mu = D_\mu - i\partial_\mu\vartheta$$
$$A_\mu \to A'_\mu = A_\mu - \tfrac{1}{e}\partial_\mu\vartheta$$

$$F_{\mu\nu} \equiv [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F_{\mu\nu} \to F'_{\mu\nu} = F_{\mu\nu}$$

$$\mathcal{L} = \overline{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$\psi(\vec{x},t) \to \psi'(\vec{x},t) = e^{i\vartheta_{\mathrm{a}}\mathbf{t}_{\mathrm{a}}}\psi(\vec{x},t)$$
$$\overline{\psi}(\vec{x},t) \to \psi'(\vec{x},t) = \overline{\psi}(\vec{x},t)e^{-i\vartheta_{\mathrm{a}}\mathbf{t}_{\mathrm{a}}}$$

$$\partial_\mu \to D_\mu = \partial_\mu + igW_{\mu,\mathrm{a}}\mathbf{t}_{\mathrm{a}}$$
$$D_\mu \to D'_\mu = D_\mu + i\left[\vartheta_{\mathrm{a}}\mathbf{t}_{\mathrm{a}}, D_\mu\right]$$
$$W_\mu \to W'_\mu = W_\mu + i\left[\vartheta_{\mathrm{a}}\mathbf{t}_{\mathrm{a}}, W_{\mu,\mathrm{a}}\mathbf{t}_{\mathrm{a}}\right] - \tfrac{1}{g}\partial_\mu\left(\vartheta_{\mathrm{a}}\mathbf{t}_{\mathrm{a}}\right)$$
$$W_{\mu\nu} \equiv [D_\mu, D_\nu] = \partial_\mu W_\nu - \partial_\nu W_\mu + ig\left[W_\mu, W_\nu\right]$$
$$W_{\mu\nu} \to W'_{\mu\nu} = W_{\mu\nu} + i\left[\vartheta_{\mathrm{a}}\mathbf{t}_{\mathrm{a}}, W_{\mu\nu}\right]$$

$$\mathcal{L} = \overline{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi - \tfrac{1}{4}W_{\mathrm{a}\mu\nu}W^{\mathrm{a}\mu\nu}$$

# Concluding remarks

- Reprise of Lagrange formalism.
- Requirement of local gauge symmetry leads to coupling structure of QED.
- Extension to more complex symmetry operations will reveal non-trivial and unique coupling structure of the SM and thus describe all known fundamental interactions.
- Next lecture on layout of the electroweak sector of the SM, from the non-trivial phenomenology to the theory.
- Prepare "*The Higgs Boson Discovery at the Large Hadron Collider*" Section 2.2.

# Backup