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From Lagrangian Density to Observable

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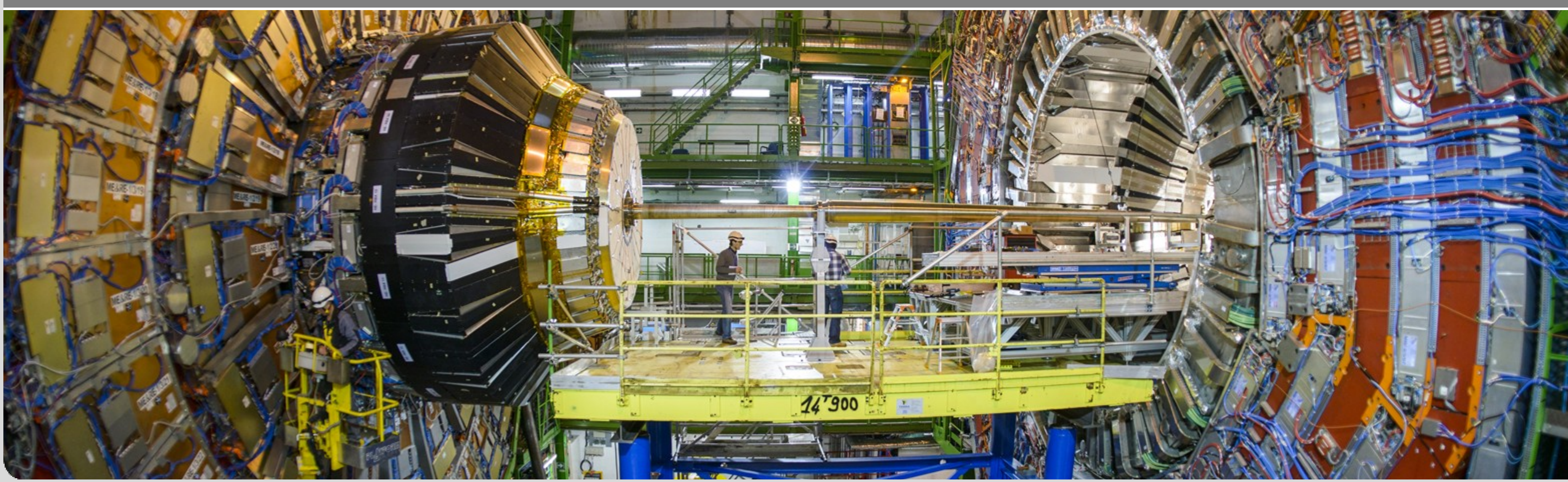
INSTITUTE OF EXPERIMENTAL PARTICLE PHYSICS (IEKP) – PHYSICS FACULTY



Schedule for today

- What is a propagator?
- Is the following statement true: “the perturbative series is a *Taylor* expansion”?

3 Perturbative series

2 Introduction of the propagator

1 Review of the QM model of scattering

Lagrangian Density → Observable

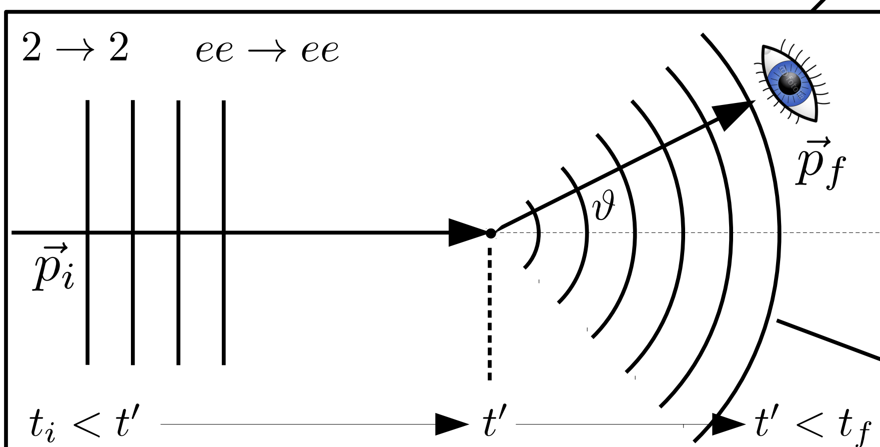
$\mathcal{L}$

QM model of particle scattering

- Consider incoming collimated beam of projectile particles on a target particle:

Scattering matrix $\mathcal{S}$ transforms initial state wave function ϕ_i into scattering wave ψ_{scat} ($\psi_{\text{scat}} = \mathcal{S} \cdot \phi_i$).

Observation (in $\Delta\Omega$): projection of plain wave ϕ_f out of spherical scattering wave ψ_{scat}.



Spherical scattering wave ψ_{scat}.

Initial particle: described by plain wave ϕ_i.

Localized potential.

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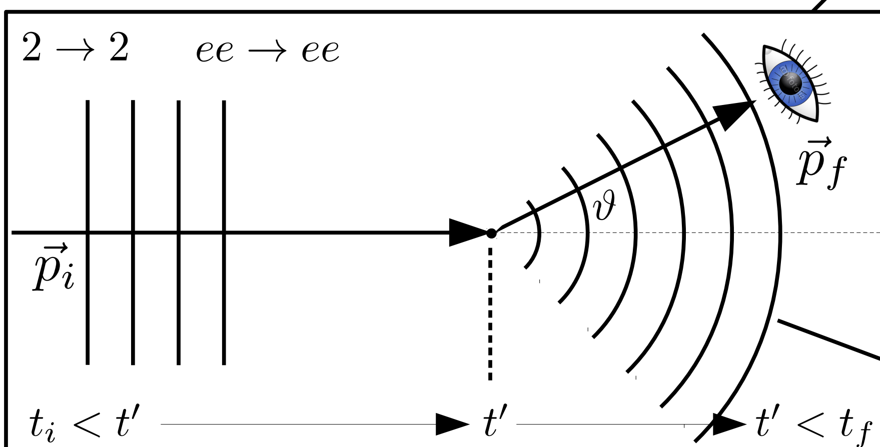
Observation probability:

$$\mathcal{S}_{fi} = \phi_f^\dagger \cdot \psi_{\text{scat}} = \phi_f^\dagger \cdot \mathcal{S} \cdot \phi_i$$



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Solution for $\psi_{\rm scat}$

- In the case of fermion scattering the scattering wave $\psi_{\rm scat}$ is obtained as a solution of the inhomogeneous *Dirac* equation for an interacting field:

$$(i\gamma^\mu \partial_\mu - m)\,\psi_{\rm scat} = -e\gamma^\mu A_\mu \psi_{\rm scat} \quad (+)$$

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- The inhomogeneous *Dirac* equation is analytically not solvable. A formal solution can be obtained by the *Green's* Function $K(x-x')$:

$$\left(i\gamma^\mu\partial_\mu - m\right) K(x-x') = \delta^4(x-x')$$

$$\psi_{\mathrm{scat}}(x) = -e\int K(x-x')\gamma^\mu A_\mu(x')\psi_{\mathrm{scat}}(x')\mathrm{d}^4x'$$

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$$(i\gamma^\mu \partial_\mu - m)\,\psi_{\text{scat}}(x) = -e \int \underbrace{(i\gamma^\mu \partial_\mu - m)\,K(x - x')}_{\delta^4(x - x')}\gamma^\mu A_\mu(x')\psi_{\text{scat}}(x')\mathrm{d}^4x'$$

$$= -e\gamma^\mu A_\mu(x)\psi_{\text{scat}}(x)$$

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- This is not a solution to (+), since ψ_{scat} appears on the left- and on the right-hand side of the equation. It turns the differential equation into an integral equation. It propagates the solution from the point x' to x.

Green's function in *Fourier* space

- The best way to find the *Green's* function is to go to the *Fourier* space:

$$K(x-x') = (2\pi)^{-4} \int \tilde{K}(p) e^{-ip(x-x')} \mathrm{d}^4 p$$ (*Fourier* transform)

Applying the *Dirac* equation to the *Fourier* transform of $K(x-x')$ turns the derivative into a product operator:

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From the uniqueness of the *Fourier* transformation the solution for $\tilde{K}(p)$ follows:

$$(\gamma^\mu p_\mu - m)\,\tilde{K}(p) = \mathbb{I}_4$$

Fermion propagator

- The *Fourier* transform of the *Green's* function is called *fermion* propagator:

$$(\gamma^\mu p_\mu - m)\, \tilde{K}(p) = \mathbb{I}_4$$

$$(\gamma^\mu p_\mu + m) \cdot (\gamma^\mu p_\mu - m)\, \tilde{K}(p) = (\gamma^\mu p_\mu + m) \cdot \mathbb{I}_4$$

$$\boxed{\tilde{K}(p) = \frac{(\gamma^\mu p_\mu + m)}{p^2 - m^2}}$$

(fermion propagator)

- The fermion propagator is a 4×4 matrix, which acts in the *Spinor* space.
- It is only defined for virtual fermions since $p^2 - m^2 = E^2 - \vec{p}^2 - m^2 \neq 0$.

Fermion propagator ↔ *Green*'s function

- The *Green's* function can be obtained from the propagator by inverse *Fourier* transformation:

$$K(x-x') = (2\pi)^{-4} \int \mathrm{d}^3\vec{p}\, e^{i\vec{p}(\vec{x}-\vec{x}')} \int_{-\infty}^{+\infty} \mathrm{d}p_0 \frac{(\gamma^\mu p_\mu + m)}{(p_0 - E)(p_0 + E)}\, e^{-ip_0(t-t')}$$

$$E = \sqrt{\vec{p}^2 + m^2}$$

- This integral can be solved with the methods of *function theory*.

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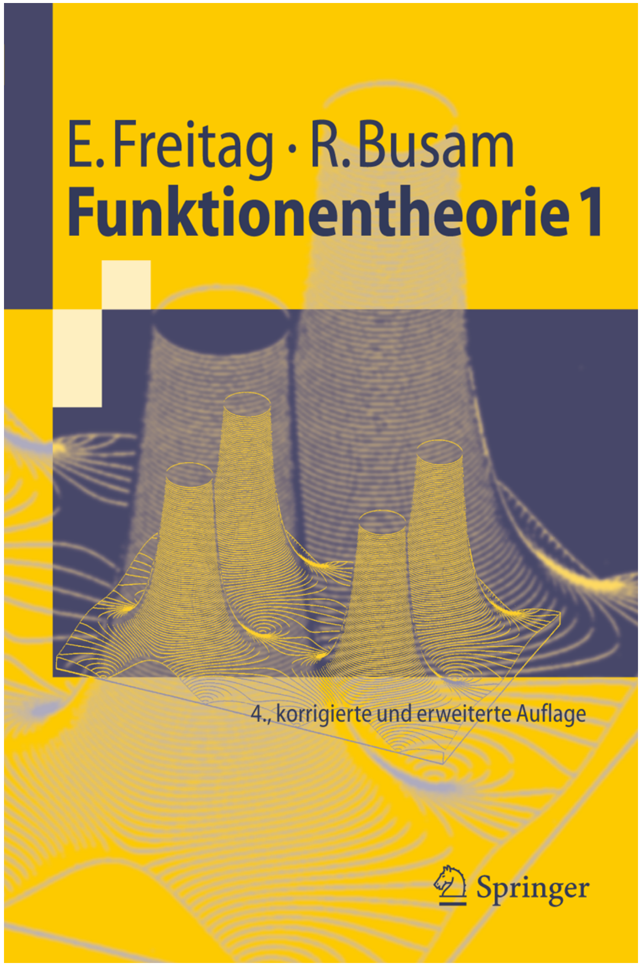
- This integral can be solved with the methods of *function theory*.
- $K(x-x')$ has two poles in the integration plane (at $p_0 = \pm E$).

Excursion into function theory

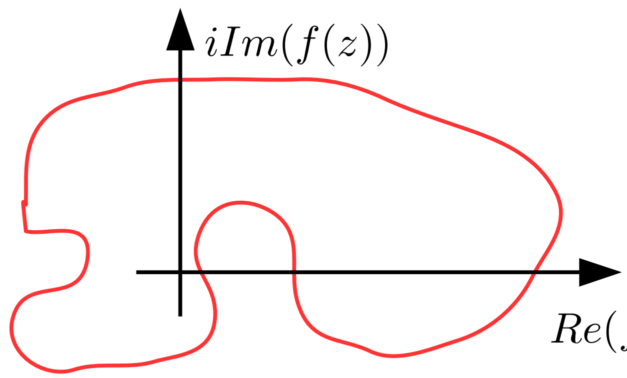


cf. Freitag/Busam Funktionentheorie

Residual theorem

- When integrating a "well behaved" function w/o poles in the complex plain the path integral along any closed path $\mathcal{C}$ is 0:

  Example: $\oint_{\mathcal{C}} z^2 \mathrm{d}z = 0$



- When integrating a "well behaved" function w/ poles in the complex plain the solution is $2\pi i \times$ the sum of "residuals" of the poles surrounded by the path:

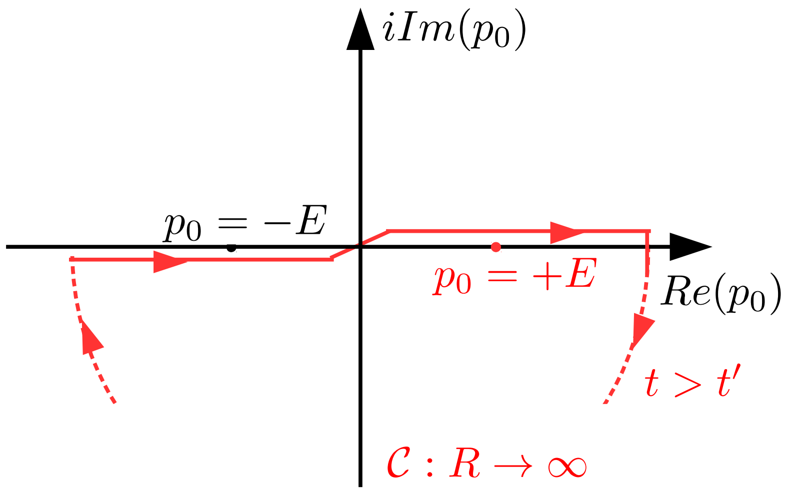
  Example: $\oint_{\mathcal{C}} \frac{R}{z} \mathrm{d}z = 2\pi i \times R$

  No matter how $\mathcal{C}$ is chosen, as long as it includes $z = (0 + i0)$.

The *Green*'s function (time integration for $t > t'$)

- Choose path $\mathcal{C}$ in complex plain to circumvent poles:

$$\int_{-\infty}^{+\infty} \mathrm{d}p_0 \frac{(\gamma^\mu p_\mu + m)}{(p_0 - E)(p_0 + E)}\, e^{-ip_0(t-t')}$$



- For $t > t'$ ($e^{-ip_0(t-t')} \to 0$ for $Im(p_0) \ll 0$):
  $\rightarrow$ close contour in lower plane & calculate integral from residual of enclosed pole.

$$\oint_{\mathcal{C}} \mathrm{d}p_0 \underbrace{\frac{1}{p_0 - E}}_{} \cdot \underbrace{\frac{(\gamma^\mu p_\mu + m)}{p_0 + E}\, e^{-ip_0(t-t')}}_{} = -2\pi i \cdot f(p_0)\big|_{p_0=+E}$$

pole at: $p_0 = +E$

residual: $f(p_0)$

Sign due to sense of integration.

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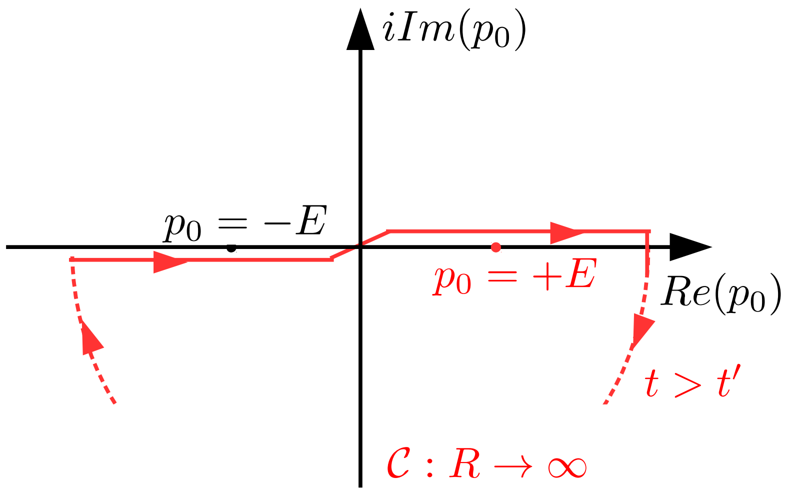

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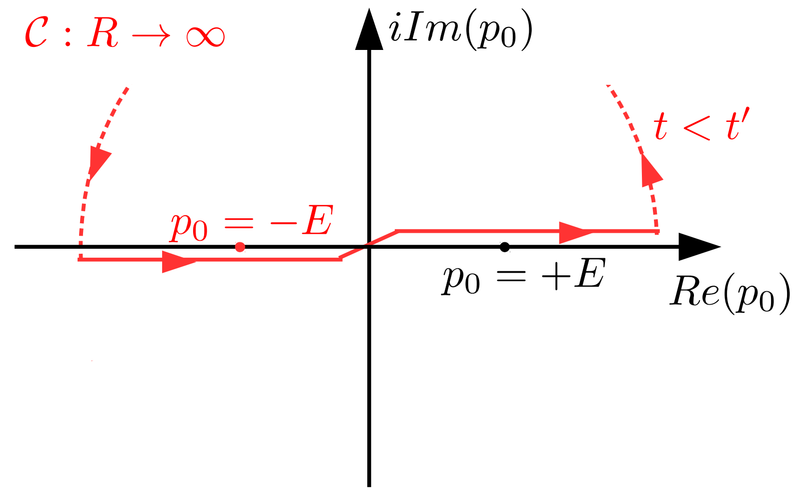

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- For $t < t'$ ($e^{+ip_0(t-t')} \to 0$ for $Im(p_0) \gg 0$):
  $\rightarrow$ close contour in upper plane & calculate integral from residual of enclosed pole.

$$\oint_{\mathcal{C}} \mathrm{d}p_0 \underbrace{\frac{1}{p_0 + E}}_{\text{pole at: } p_0 = -E} \cdot \underbrace{\frac{(\gamma^\mu p_\mu + m)}{p_0 - E}\, e^{-ip_0(t-t')}}_{\text{residual: } f(p_0)} = +2\pi i \cdot f(p_0)\big|_{p_0 = -E}$$

pole at: $p_0 = -E$

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Sign due to sense of integration.

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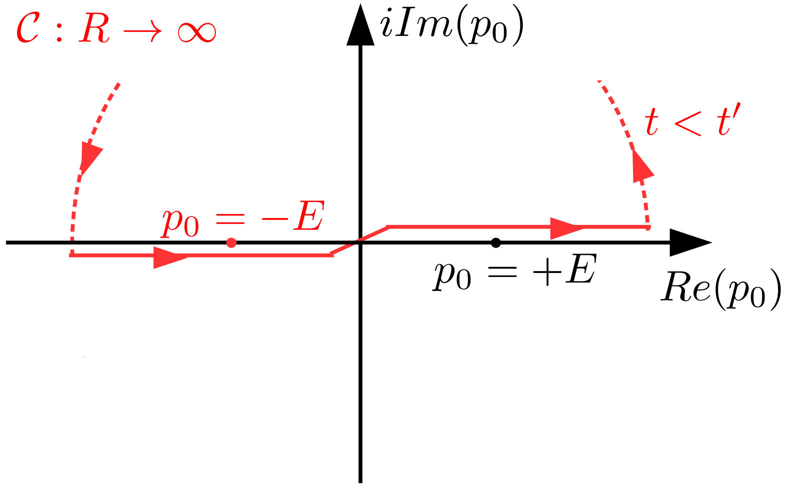

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$$K(x - x') = -i(2\pi)^{-3} \int \mathrm{d}^3\vec{p} \, \frac{-\gamma^0 E - \vec{\gamma}\vec{p} + m}{2E} \cdot e^{+iE(t-t')+i\vec{p}(\vec{x}-\vec{x}')}$$

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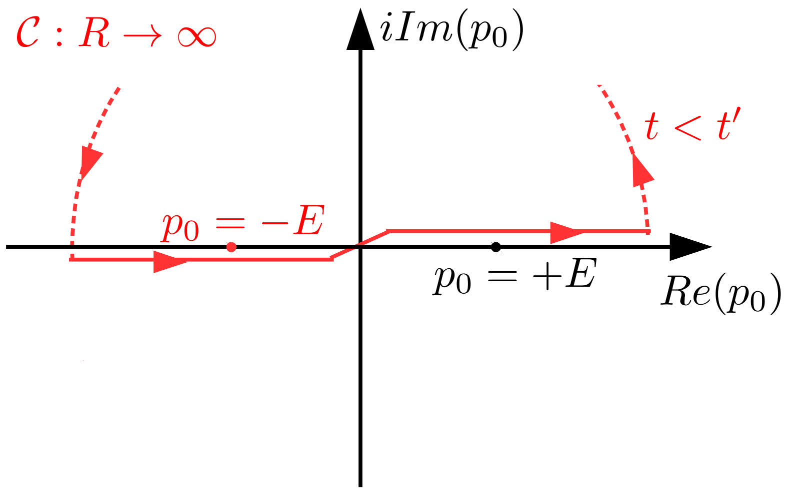

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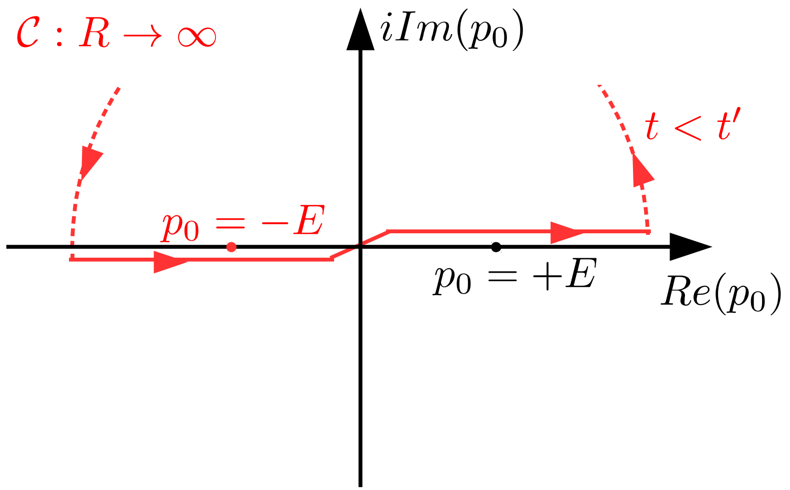
$$K(x - x') = -i(2\pi)^{-3} \int \mathrm{d}^3\vec{p}\, \frac{-\gamma^0 E - \vec{\gamma}\vec{p} + m}{2E} \cdot e^{+iE(t-t') + i\vec{p}(\vec{x} - \vec{x}')}$$

Sign due to $p_0 = -E$ in integral kernel.

The *Green*'s function (Nota Bene)

- Choose path $\mathcal{C}$ in complex plain to circumvent poles:

$$\int_{-\infty}^{+\infty} \mathrm{d}p_0 \frac{(\gamma^\mu p_\mu + m)}{(p_0 - E)(p_0 + E)} \, e^{-ip_0(t-t')}$$



- The bending of the integration path can be avoided by shifting the poles by ϵ.

$$\left[p_0 + \left(E - \frac{i\epsilon}{2E}\right)\right] \cdot \left[p_0 - \left(E - \frac{i\epsilon}{2E}\right)\right] = p_0^2 - \left(\vec{p}^2 + m^2\right) + i\epsilon$$

$$= p^2 - m^2 + i\epsilon$$

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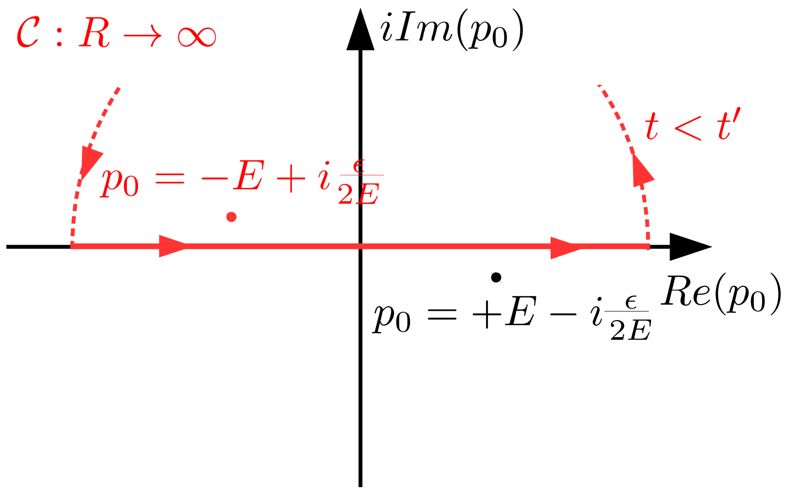

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$$= p^2 - m^2 + i\epsilon$$

$\left(-E + i\frac{\epsilon}{2E}\right)$ $\quad$ $\left(+E - i\frac{\epsilon}{2E}\right)$

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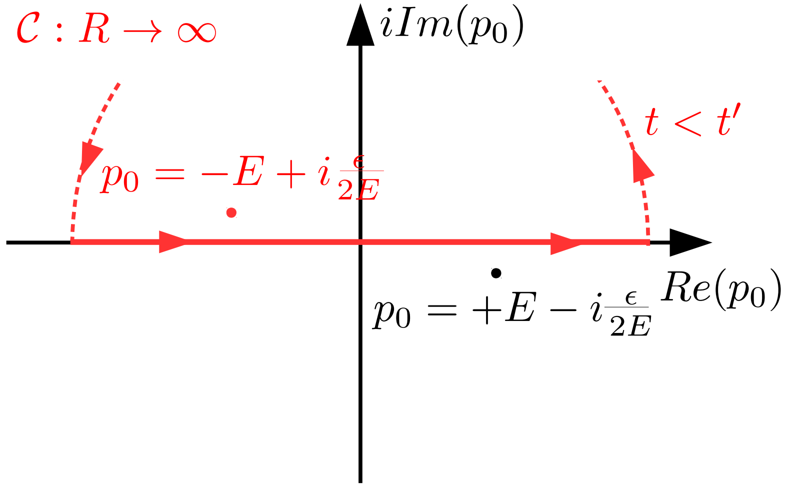

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$$= p^2 - m^2 + i\epsilon$$

$$\tilde{K}(p) = \frac{(\gamma^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon} \qquad \epsilon > 0$$

(fermion propagator)

Summary of time evolution

$$\tilde{K}(p) = \frac{(\gamma^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon} \qquad \epsilon > 0$$

(Fermion propagator in momentum space)

- *Green's* function (for $t > t'$, forward evolution):

$$K(x - x') = -i(2\pi)^{-3} \int \mathrm{d}^3\vec{p}\, \frac{+\gamma^0 E - \vec{\gamma}\vec{p} + m}{2E} \cdot e^{-iE(t-t')+i\vec{p}(\vec{x}-\vec{x}')}$$

- *Green's* function (for $t < t'$, backward evolution):

$$K(x - x') = -i(2\pi)^{-3} \int \mathrm{d}^3\vec{p}\, \frac{-\gamma^0 E - \vec{\gamma}\vec{p} + m}{2E} \cdot e^{+iE(t-t')+i\vec{p}(\vec{x}-\vec{x}')}$$

- But why did I choose explicitly THIS integration path and not another one?

Summary of time evolution

$$\tilde{K}(p) = \frac{(\gamma^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon} \qquad \epsilon > 0$$

(Fermion propagator in momentum space)

- The chosen integration path defines the time evolution of the solution.

- General solution to (inhomogeneous) Dirac equation:

$$\phi(t,\vec{x}) = \begin{cases} i\int \mathrm{d}^3\vec{x}' K(x-x')\gamma^0\phi(t',\vec{x}') & \text{for } t > t' \\ 0 & \text{for } t < t' \end{cases}$$

particle w/ pos. energy traveling forward in time.

$$\overline{\phi}(t,\vec{x}) = \begin{cases} 0 & \text{for } t > t' \\ i\int \mathrm{d}^3\vec{x}' \overline{\phi}(t',\vec{x}')\gamma^0 K(x-x') & \text{for } t < t' \end{cases}$$

particle w/ pos. energy traveling backward in time.

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particle w/ neg. energy traveling forward in time.

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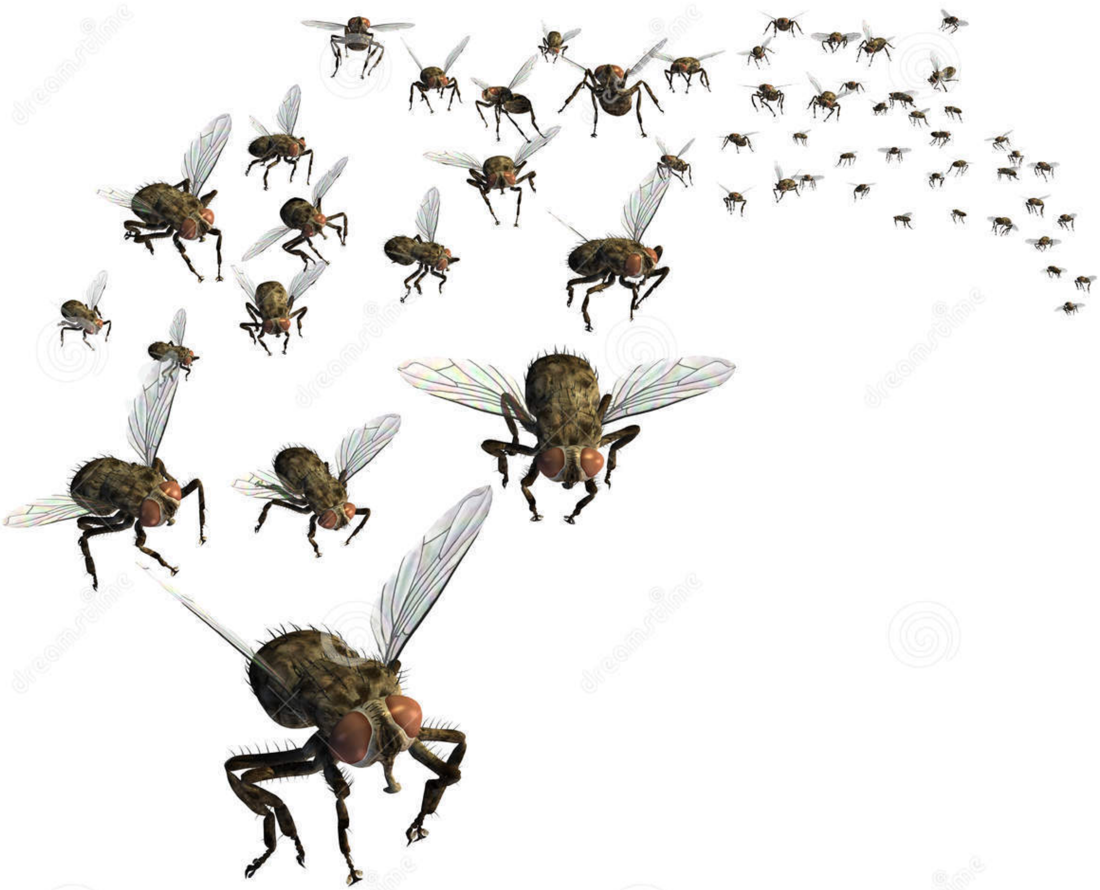
particle w/ neg. energy traveling backward in time.

The perturbative series



The perturbative series

- The integral equation can be solved iteratively:

$$\psi_{\mathrm{scat}}(x) = \phi(x) - e\int K(x-x')\gamma^{\mu}A_{\mu}(x')\psi_{\mathrm{scat}}(x')\mathrm{d}^4x'$$

($\phi(x)$ = solution of the homogeneous *Dirac* equation)

- 0th order perturbation theory:

$$\psi^{(0)}(x) = \phi(x)$$

- Just take $\phi(x)$ as solution ($\rightarrow$ boring).

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- 1st order perturbation theory:

$$\psi^{(1)}(x) = \psi^{(0)}(x) - e\int K(x-x')\gamma^\mu A_\mu(x')\psi^{(0)}(x')\mathrm{d}^4x'$$

- Assume that $\psi^{(0)}(x)$ is close enough to actual solution on RHS.

The perturbative series

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- Assume that $\psi^{(0)}(x)$ is close enough to actual solution on RHS.

- 2nd order perturbation theory:

$$\psi^{(2)}(x) = \psi^{(0)}(x) - e\int K(x-x')\gamma^\mu A_\mu(x')\psi^{(1)}(x')\mathrm{d}^4x'$$

- Take $\psi^{(1)}(x)$ as better approximation at RHS to solve inhomogeneous equation.

The perturbative series

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- The integral equation can be solved iteratively:

$$\psi_{\text{scat}}(x) = \phi(x) - e\int K(x-x')\gamma^\mu A_\mu(x')\psi_{\text{scat}}(x')\mathrm{d}^4x'$$

($\phi(x)$ = solution of the homogeneous *Dirac* equation)

- 0th order perturbation theory:

$$\psi^{(0)}(x) = \phi(x)$$

- Just take $\phi(x)$ as solution (→ boring).

- 1st order perturbation theory:

$$\psi^{(1)}(x) = \psi^{(0)}(x) - e\int K(x-x')\gamma^\mu A_\mu(x')\psi^{(0)}(x')\mathrm{d}^4x'$$

- Assume that $\psi^{(0)}(x)$ is close enough to actual solution on RHS.

- 2nd order perturbation theory:

$$\psi^{(2)}(x) = \psi^{(0)}(x) - e\int K(x-x')\gamma^\mu A_\mu(x')\psi^{(0)}(x')\mathrm{d}^4x' + e^2\iint K(x-x')\gamma^\mu A_\mu(x')K(x'-x'')\gamma^\mu A_\mu(x'')\psi^{(0)}(x'')\mathrm{d}^4x'\mathrm{d}^4x''$$

The perturbative series

- The integral equation can be solved iteratively:

$$\psi_{\rm scat}(x) = \phi(x) - e\int K(x-x')\gamma^\mu A_\mu(x')\psi_{\rm scat}(x')\mathrm{d}^4x'$$

($\phi(x)$ = solution of the homogeneous *Dirac* equation)

- 0[th] order perturbation theory:

$$\psi^{(0)}(x) = \phi(x)$$

- Just take $\phi(x)$ as solution (→ boring).

- 1[st] order perturbation theory:

$$\psi^{(1)}(x) = \psi^{(0)}(x) - e\int K(x-x')\gamma^\mu A_\mu(x')\psi^{(0)}(x')\mathrm{d}^4x'$$

- Assume that $\psi^{(0)}(x)$ is close enough to actual solution on RHS.

- 2[nd] order perturbation theory:

$$\psi^{(2)}(x) = \psi^{(0)}(x) - e\int K(x-x')\gamma^\mu A_\mu(x')\psi^{(0)}(x')\mathrm{d}^4x' + e^2\iint K(x-x')\gamma^\mu A_\mu(x')K(x'-x'')\gamma^\mu A_\mu(x'')\psi^{(0)}(x'')\mathrm{d}^4x'\mathrm{d}^4x''$$

This procedure is justified since $e = \sqrt{4\pi\alpha} \approx \sqrt{4\pi/137} \ll 1$.

The matrix element $\mathcal{S}_{fi}$

- $\mathcal{S}_{fi}$ is obtained from the projection of the scattering wave $\psi_{\rm scat}$ on $\phi_f = \phi(x_f)$:

$$\mathcal{S}_{fi} = \int \mathrm{d}^4 x_f \phi_f^\dagger(x_f)\psi_{\rm scat}(x_f) = \int \mathrm{d}^4 x_f \phi_f^\dagger(x_f)\mathcal{S}\phi_i(x_f)$$
$$= \delta_{fi} + \mathcal{S}_{fi}^{(1)} + \mathcal{S}_{fi}^{(2)} + \dots$$

"LO" ($\mathcal{S}_{fi}^{(1)}$) "NLO" ($\mathcal{S}_{fi}^{(2)}$)

- 1st order perturbation theory:

$$\mathcal{S}_{fi}^{(1)} = -e\int \mathrm{d}^4 x' \underbrace{\int \mathrm{d}^4 x_f \phi_f^\dagger(x_f) K(x_f - x')}_{\equiv -i\overline{\phi}_f(x') = -i\overline{\phi}(x_f)} \overbrace{K(x_f - x')\gamma^\mu A_\mu(x')\phi_i(x')}^{\equiv \phi_f(x_f) = \phi(x_f)}$$

For $E > 0$ and $t_f > t'$ respectively.

$$\phi(x_f) = \; -e\int \mathrm{d}^4 x' K(x_f - x')\gamma^\mu A_\mu(x')\phi(x')$$ cf. slide 7

$$\overline{\phi}(x') = i\int \mathrm{d}^3\vec{x}\,\overline{\phi}(x_f)\gamma^0 K(x' - x_f) = -i\int \mathrm{d}^3\vec{x}_f\overline{\phi}(x_f)\gamma^0 K(x_f - x')$$ cf. slide 28

NB: the time integration has already been carried out for the backward evolution from t_f to t' to arrive at the equation of slide 28.

The matrix element $\mathcal{S}_{fi}$

- $\mathcal{S}_{fi}$ is obtained from the projection of the scattering wave ψ_{scat} on $\phi_f = \phi(x_f)$:

$$\mathcal{S}_{fi} = \int \mathrm{d}^4 x_f \phi_f^\dagger(x_f)\psi_{\text{scat}}(x_f) = \int \mathrm{d}^4 x_f \phi_f^\dagger(x_f)\mathcal{S}\phi_i(x_f)$$
$$= \delta_{fi} + \mathcal{S}_{fi}^{(1)} + \mathcal{S}_{fi}^{(2)} + \dots$$

"LO" ($\mathcal{S}_{fi}^{(1)}$) "NLO" ($\mathcal{S}_{fi}^{(2)}$)

- 1st order perturbation theory:

$$\mathcal{S}_{fi}^{(1)} = -e \int \mathrm{d}^4 x' \int \mathrm{d}^3 x_f \phi_f^\dagger(x_f) K(x_f - x')\gamma^\mu A_\mu(x')\phi_i(x')$$

$$\mathcal{S}_{fi}^{(1)} = i \cdot \int \mathrm{d}^4 x' e\, \overline{\phi}_f(x')\gamma^\mu A_\mu(x')\phi_i(x') \qquad \text{(1st order matrix element)}$$

This corresponds exactly to the IA term in $\mathcal{L}$, including the multiplication by i (cf. Lecture-05 slide 39).

Concluding Remarks

- Amplitude of scattering processes can be obtained from a QM model via perturbation theory.
- Introduced propagator as formal solution of the equation of motion for fermion case.
- Derived 1st order matrix element.

We are not yet done: since projectile is back-scattered $A_\mu(x')$ also evolves! This part will be discussed during the next lecture.

$$\mathcal{S}_{fi}^{(1)} = i \cdot \int \mathrm{d}^4x' e\, \overline{\phi}_f(x')\gamma^\mu A_\mu(x')\phi_i(x') \qquad \text{(1}^{\text{st}}\text{ order matrix element)}$$

- In the next lecture we will complete the picture of Feynman rules for the simple example of electron scattering.

Backup