

From Lagrangian Density to Observable

Roger Wolf
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INSTITUTE OF EXPERIMENTAL PARTICLE PHYSICS (IEKP) – PHYSICS FACULTY



Schedule for today

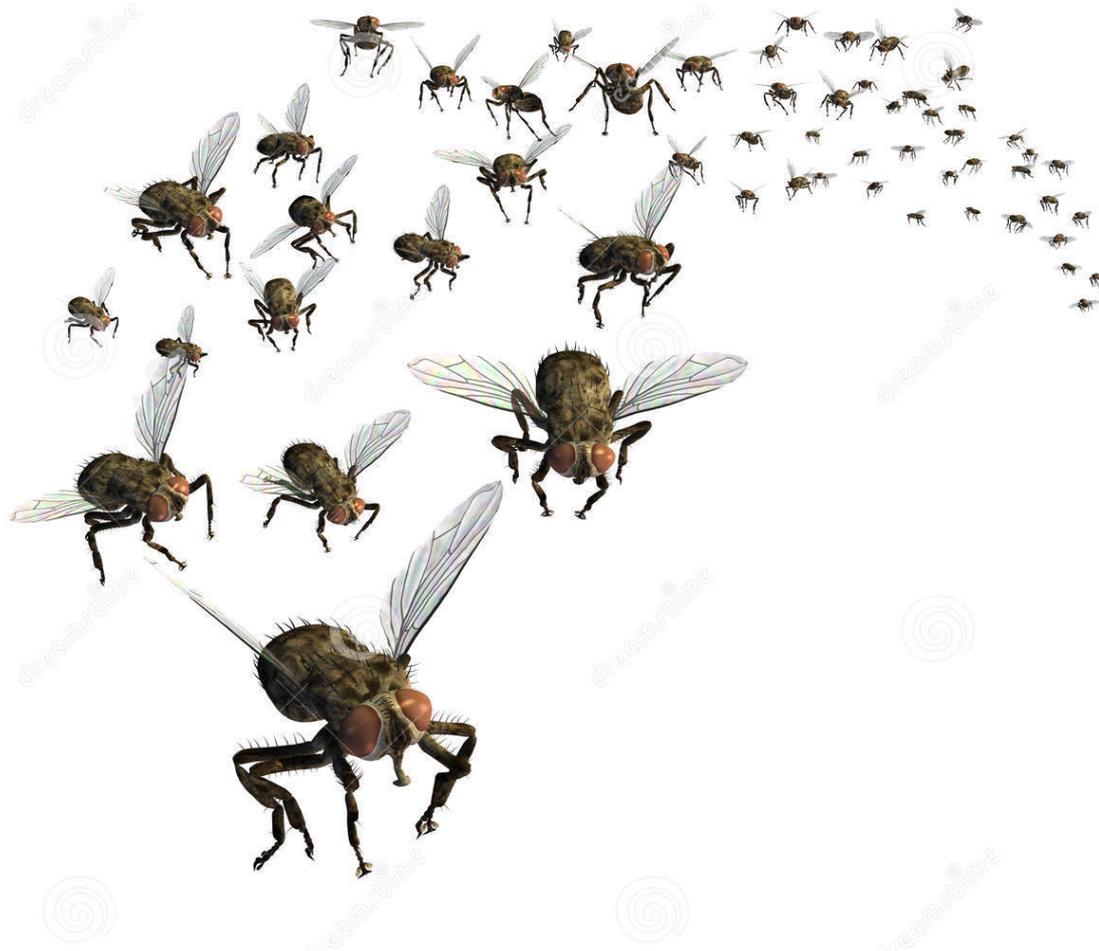
- Does a Feynman diagram have a time direction? If yes, what is it?

3 Intrinsic bounds on the Higgs boson mass in the SM

2 Discussion of higher order effects in perturbation theory

1 Completion of cross section calculation

The perturbative series



- The integral equation can be solved iteratively:

$$\psi_{\text{scat}}(x) = \phi(x) - e \int K(x - x') \gamma^\mu A_\mu(x') \psi_{\text{scat}}(x') d^4x'$$

- 0th order perturbation theory:

$$\psi^{(0)}(x) = \phi(x)$$

- 1st order perturbation theory:

$$\begin{aligned} \psi^{(1)}(x) &= \psi^{(0)}(x) \\ &\quad - e \int K(x - x') \gamma^\mu A_\mu(x') \psi^{(0)}(x') d^4x' \end{aligned}$$

- 2nd order perturbation theory:

$$\begin{aligned} \psi^{(2)}(x) &= \psi^{(0)}(x) \\ &\quad - e \int K(x - x') \gamma^\mu A_\mu(x') \psi^{(1)}(x') d^4x' \end{aligned}$$

($\phi(x)$ = solution of the homogeneous Dirac equation)

- Just take $\phi(x)$ as solution (\rightarrow boring).
- Assume that $\psi^{(0)}(x)$ is close enough to actual solution on RHS.
- Take $\psi^{(1)}(x)$ as better approximation at RHS to solve inhomogeneous equation.

- The integral equation can be solved iteratively:

$$\psi_{\text{scat}}(x) = \phi(x) - e \int K(x - x') \gamma^\mu A_\mu(x') \psi_{\text{scat}}(x') d^4x'$$

($\phi(x)$ = solution of the homogeneous Dirac equation)

- 0th order perturbation theory:

$$\psi^{(0)}(x) = \phi(x)$$

- 1st order perturbation theory:

$$\begin{aligned} \psi^{(1)}(x) &= \psi^{(0)}(x) \\ &\quad - e \int K(x - x') \gamma^\mu A_\mu(x') \psi^{(0)}(x') d^4x' \end{aligned}$$

- 2nd order perturbation theory:

$$\begin{aligned} \psi^{(2)}(x) &= \psi^{(0)}(x) \\ &\quad - e \int K(x - x') \gamma^\mu A_\mu(x') \psi^{(0)}(x') d^4x' \\ &\quad + e^2 \iint K(x - x') \gamma^\mu A_\mu(x') K(x' - x'') \gamma^\mu A_\mu(x'') \psi^{(0)}(x'') d^4x' d^4x'' \end{aligned}$$

- Just take $\phi(x)$ as solution (\rightarrow boring).
- Assume that $\psi^{(0)}(x)$ is close enough to actual solution on RHS.

The matrix element \mathcal{S}_{fi}

- \mathcal{S}_{fi} is obtained from the projection of the scattering wave ψ_{scat} on $\phi_f = \phi(x_f)$:

$$\mathcal{S}_{fi} = \int d^4x_f \phi_f^\dagger(x_f) \psi_{\text{scat}}(x_f) = \int d^4x_f \phi_f^\dagger(x_f) \mathcal{S} \phi_i(x_f)$$

$$= \delta_{fi} + \mathcal{S}_{fi}^{(1)} + \mathcal{S}_{fi}^{(2)} + \dots$$

“LO”

“NLO”

- 1st order perturbation theory: $\equiv \phi_f(x_f) = \phi(x_f)$
- $$\mathcal{S}_{fi}^{(1)} = -e \int d^4x' \int d^4x_f \phi_f^\dagger(x_f) \underbrace{K(x_f - x') \gamma^\mu A_\mu(x')}_{\equiv -i\bar{\phi}_f(x')} \phi_i(x')$$
- $$\equiv -i\bar{\phi}_f(x') = -i\bar{\phi}(x_f)$$

For $E > 0$ and $t_f > t'$ respectively.

$$\phi(x_f) = -e \int d^4x' K(x_f - x') \gamma^\mu A_\mu(x') \phi(x')$$

cf. slide 7

$$\bar{\phi}(x') = i \int d^3\vec{x} \bar{\phi}(x_f) \gamma^0 K(x' - x_f) = -i \int d^3\vec{x}_f \bar{\phi}(x_f) \gamma^0 K(x_f - x')$$

cf. slide 28

The matrix element \mathcal{S}_{fi}

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$$\begin{aligned}\mathcal{S}_{fi} &= \int d^4x_f \phi_f^\dagger(x_f) \psi_{\text{scat}}(x_f) = \int d^4x_f \phi_f^\dagger(x_f) \mathcal{S} \phi_i(x_f) \\ &= \delta_{fi} + \mathcal{S}_{fi}^{(1)} + \mathcal{S}_{fi}^{(2)} + \dots\end{aligned}$$

“LO”

“NLO”

- 1st order perturbation theory:

$$\mathcal{S}_{fi}^{(1)} = -e \int d^4x' \int d^3x_f \phi_f^\dagger(x_f) K(x_f - x') \gamma^\mu A_\mu(x') \phi_i(x')$$

$$\mathcal{S}_{fi}^{(1)} = i \cdot \int d^4x' e \bar{\phi}_f(x') \gamma^\mu A_\mu(x') \phi_i(x') \quad (1^{\text{st}} \text{ order matrix element})$$

This corresponds exactly to the IA term in \mathcal{L} , including the multiplication by i (cf. Lecture-05 slide 39).

The photon propagator

- The evolution of A_μ happens according to the inhomogeneous wave equation of the photon field (in Lorentz gauge $\partial_\mu A^\mu = 0$)

$$\square A^\mu = eJ^\mu \quad (++)$$

- We solve (++) again formally via the *Green's* function $D^{\mu\nu}(x - x')$ with the property:

$$\square D^{\mu\nu}(x - x') = g^{\mu\nu} \delta^4(x - x')$$

$$A^\mu(x) = e \int d^4x' D^{\mu\nu}(x - x') J_\nu(x')$$

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$$\square A^\mu(x) = e \int d^4x' \underbrace{\square D^{\mu\nu}(x - x')}_{g^{\mu\nu} \delta^4(x - x')} J_\nu(x') = eJ^\mu(x)$$

Green's function in *Fourier* space (fast forward)

- Check for the concrete form of the *Green's* function again first in *Fourier* space:

$$D^{\mu\nu}(x - x') = (2\pi)^{-4} \int d^4q \tilde{D}^{\mu\nu}(q) e^{-iq(x-x')} \quad (\text{Fourier transform})$$

In analogy to the fermion case the defining property of $D^{\mu\nu}(x - x')$ in *Fourier* space

$$\begin{aligned} \square D^{\mu\nu}(x - x') &= (2\pi)^{-4} \int d^4q (-q^2) \tilde{D}^{\mu\nu}(q) e^{-iq(x-x')} \stackrel{!}{=} \\ &= (2\pi)^{-4} \int d^4q g^{\mu\nu} e^{-iq(x-x')} = g^{\mu\nu} \delta^4(x - x') \end{aligned}$$

(omitting the discussion of integral paths) leads to

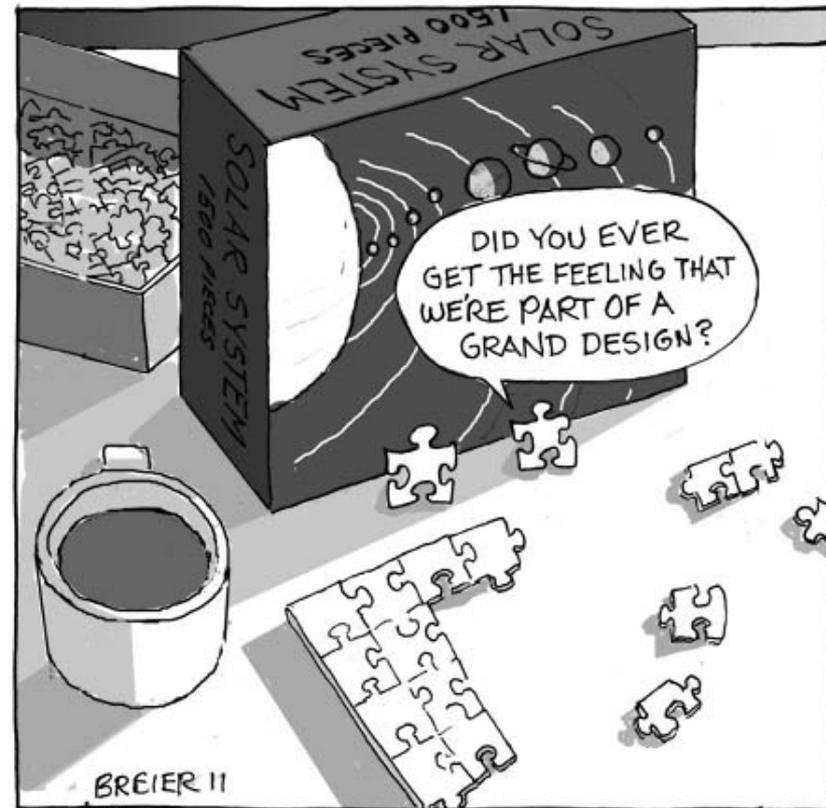
$$\boxed{\tilde{D}^{\mu\nu}(q) = \frac{-g^{\mu\nu}}{q^2 + i\epsilon} \quad \epsilon > 0} \quad (\text{photon propagator})$$

Green's function in *Fourier* space (fast forward)

- The *Green's* function can again be obtained from the inverse *Fourier* transform.

$$D^{\mu\nu}(x - x') = (2\pi)^{-4} \int d^4q \frac{-g^{\mu\nu}}{q^2 + i\epsilon} e^{-iq(x-x')}$$

- We have now collected all pieces of the puzzle to complete the cross section calculation.

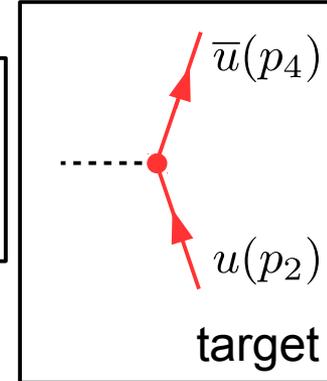


On the way to completion...

- Ansatz for target current:

$$\bar{\psi}_f(x'') = \bar{u}(p_4)e^{ip_4x''} \quad \psi_i(x'') = u(p_2)e^{-ip_2x''}$$

$$eJ^\nu(x'') = e \cdot \bar{\psi}_f(x'')\gamma^\nu\psi_i(x'') = e \cdot \bar{u}(p_4)\gamma^\nu u(p_2)e^{i(p_4-p_2)x''}$$



- Combination with photon propagator to get the evolution of A_μ :

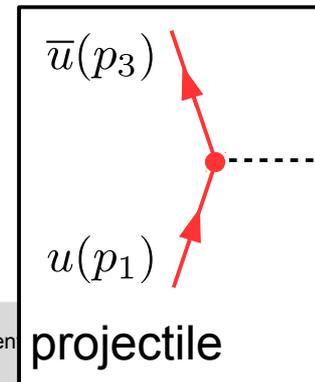
$$A_\mu(x') = e \int d^4x'' D^{\mu\nu}(x' - x'') J^\nu(x'')$$

$$= e \cdot \int d^4x'' (2\pi)^{-4} \int d^4q \frac{-g_{\mu\nu}}{q^2+i\epsilon} e^{i(p_4-p_2+q)x''} e^{-iqx'} \bar{u}(p_4)\gamma^\nu u(p_2)$$

$$= e \cdot \int d^4q \frac{-g_{\mu\nu}}{q^2+i\epsilon} \delta^4(p_4 - p_2 + q) e^{-iqx'} \bar{u}(p_4)\gamma^\nu u(p_2)$$

- Ansatz for projectile current:

$$\bar{\phi}_f(x') = \bar{u}(p_3)e^{ip_3x'} \quad \phi_i(x') = u(p_1)e^{-ip_1x'}$$



On the way to completion...

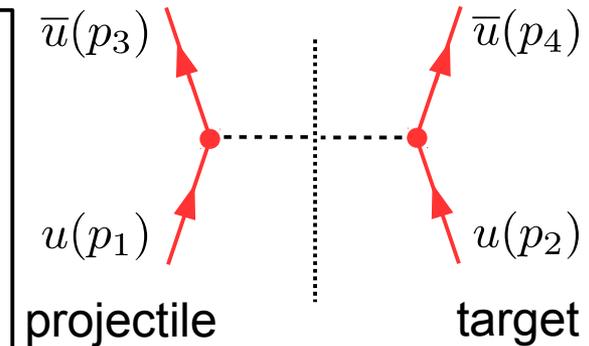
- 1st order matrix element:

$$\mathcal{S}_{fi}^{(1)} = i \cdot \int d^4x' e \bar{\phi}_f(x') \gamma^\mu A_\mu(x') \phi_i(x')$$

$$\bar{\phi}_f(x') = \bar{u}(p_3) e^{ip_3x'}$$

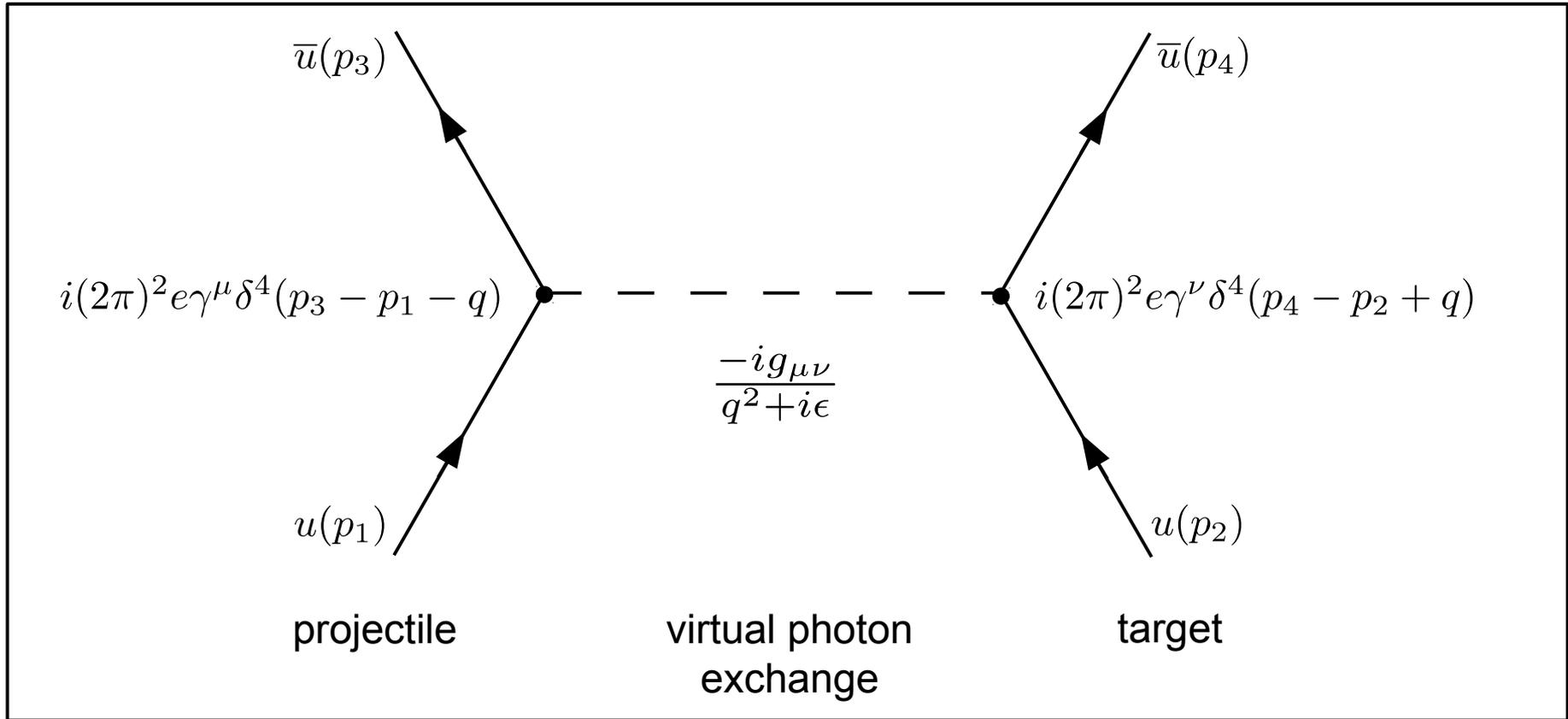
$$\phi_i(x') = u(p_1) e^{-ip_1x'}$$

$$A_\mu(x') = e \cdot \int d^4q \frac{-g_{\mu\nu}}{q^2 + i\epsilon} \delta^4(p_4 - p_2 + q) e^{-iqx'} u(p_4) \gamma^\nu u(p_2)$$



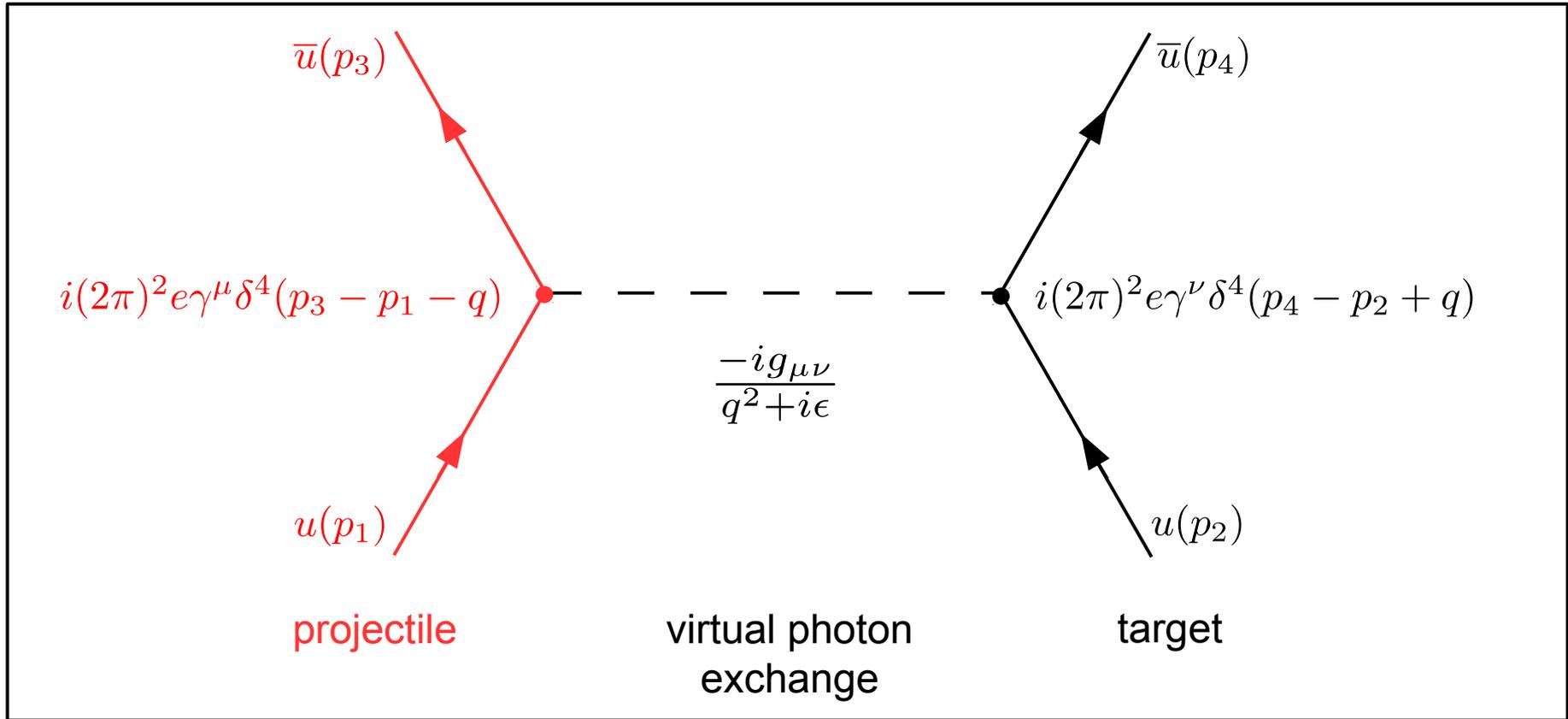
$$\begin{aligned} \mathcal{S}_{fi}^{(1)} &= ie^2 \cdot \int d^4q \underbrace{\int d^4x' e^{i(p_3 - p_1 - q)x'}}_{(2\pi)^4 \delta^4(p_3 - p_1 - q)} \bar{u}(p_3) \gamma^\mu u(p_1) \frac{-g_{\mu\nu}}{q^2 + i\epsilon} \delta^4(p_4 - p_2 + q) \bar{u}(p_4) \gamma^\nu u(p_2) \\ &= i ((2\pi)^2 e)^2 \cdot \int d^4q \delta^4(p_3 - p_1 - q) \bar{u}(p_3) \gamma^\mu u(p_1) \frac{-g_{\mu\nu}}{q^2 + i\epsilon} \delta^4(p_4 - p_2 + q) \bar{u}(p_4) \gamma^\nu u(p_2) \end{aligned}$$

The matrix element \mathcal{S}_{fi} (complete picture)



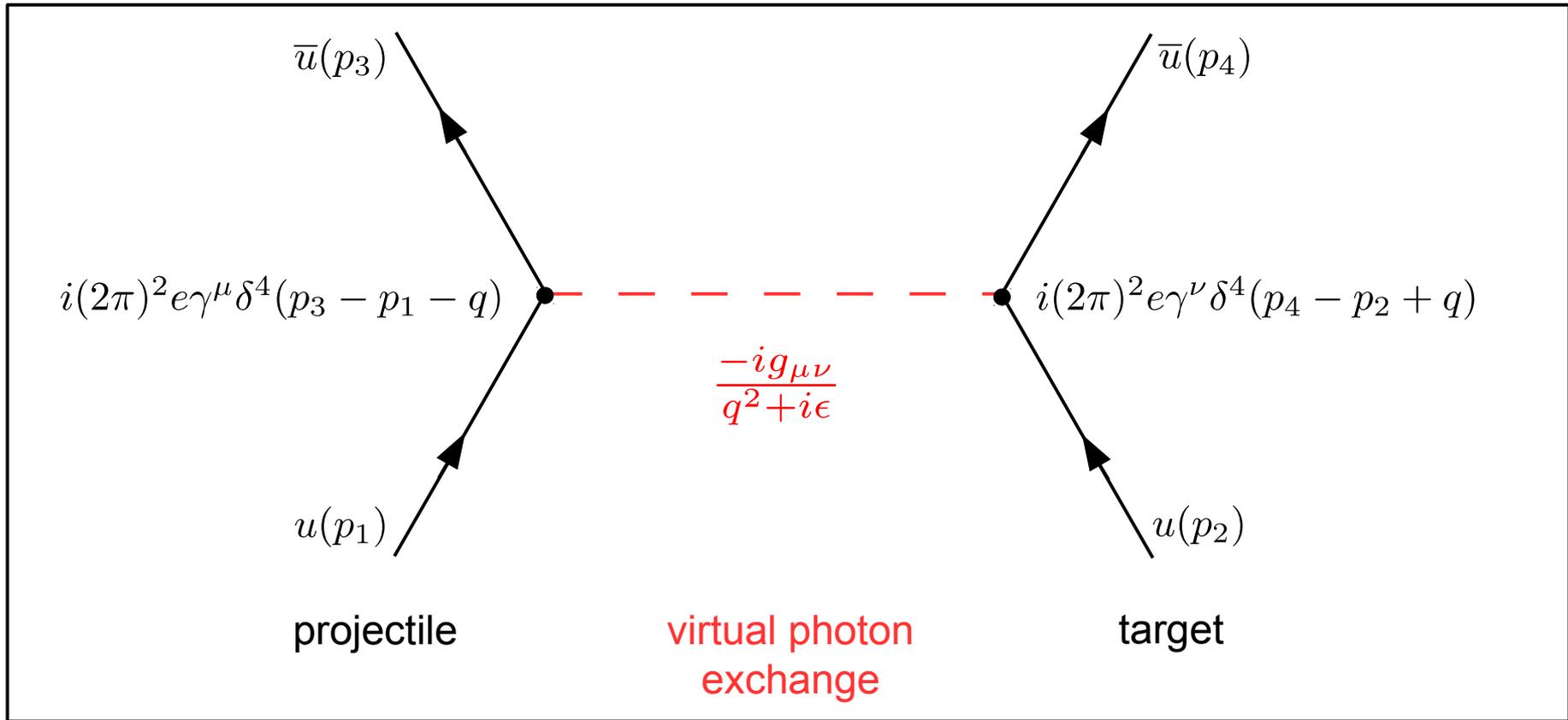
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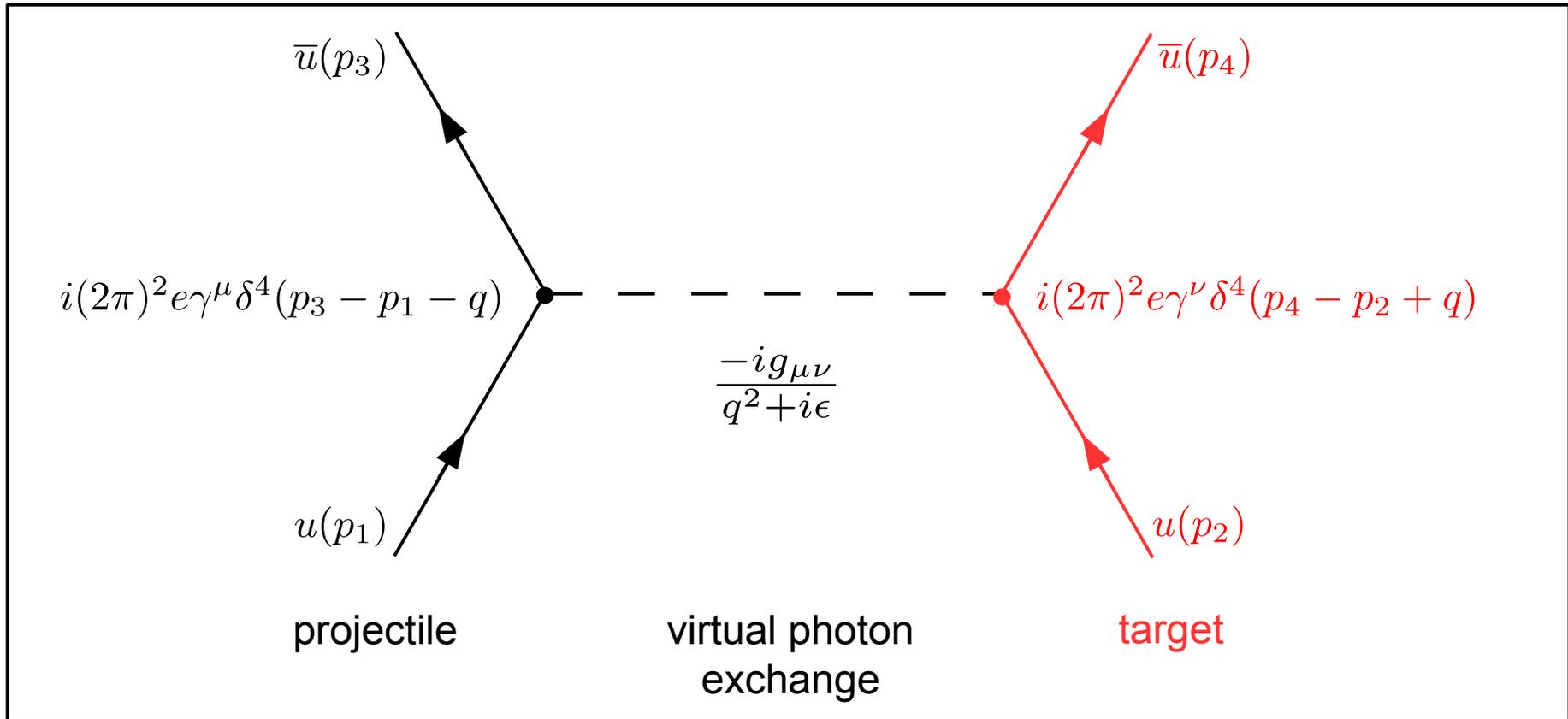
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Feynman Rules (QED)

- *Feynman* diagrams are a way to represent the elements of the matrix element calculation:

Legs:



$$u(p) \quad (\bar{u}(p))$$

- Incoming (outgoing) fermion.



$$\epsilon_\mu(k) \quad (\epsilon_\mu^*(k))$$

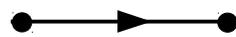
- Incoming (outgoing) photon.

Vertices:

- $i(2\pi)^2 e \gamma^\mu \cdot \delta^4(p_f - p_i - q)$

- Lepton-photon vertex.

Propagators:



$$\frac{i(\gamma^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon}$$

- Fermion propagator.



$$\frac{-ig^{\mu\nu}}{q^2 + i\epsilon}$$

- Photon propagator.

Four-momenta of all virtual particles have to be integrated out.

- *Feynman* diagrams are a way to represent the elements of the matrix element calculation:
- A *Feynman* diagram:
 - is not just a sketch, it has a **strict mathematical correspondence**.
 - is drawn in **momentum space**.
 - **does not have a time direction**. Only time information is introduced by choice of initial and final state by reader (e.g. t-channel vs s-channel processes).

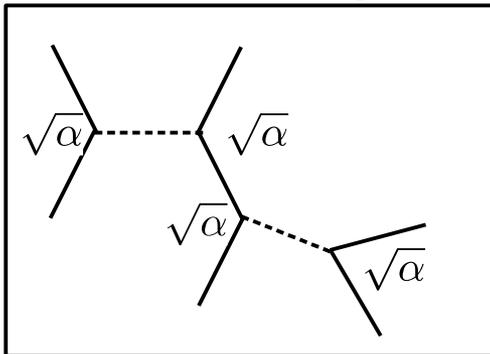


- Scattering amplitude \mathcal{S}_{fi} only known in perturbation theory.
- Works **better the smaller the perturbation** is:
 - QED: $\alpha \approx \frac{1}{137}$
 - QFD: $\alpha_w = \frac{\alpha}{\sin^2(\theta_W)} \approx 4 \cdot \alpha_{em}$ with $\theta_W = 28.74^\circ$
 - QCD: $\alpha_s(m_Z) \approx 0.12$
- If perturbation theory works well, the first contribution of the scattering amplitude is already sufficient to describe the main features of the scattering process.
- This contribution is of order " α ". It is often called *Tree Level*, *Born Level* or *Leading Order* (LO) scattering amplitude.
- Any higher order of the scattering amplitude in perturbation theory appears at higher orders of " α ".

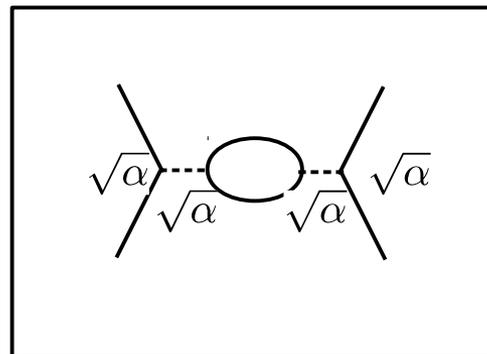
Order α^2 diagrams (QED)

- We have only discussed contributions to \mathcal{S}_{fi} , which are of order α^1 in QED. (e.g. LO $ee \rightarrow ee$ scattering) .
- Diagrams which **contribute to order α^2** would look like this:

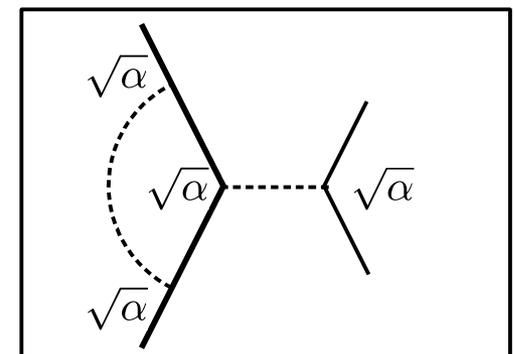
Additional legs:



Loops:



(in propagators or legs)

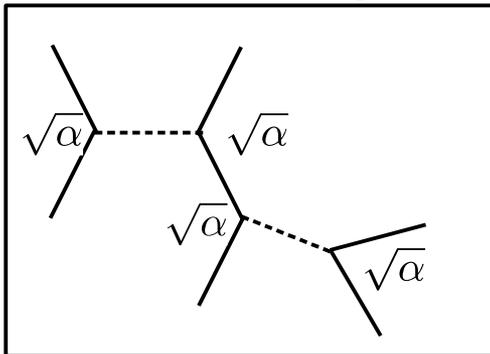


(in vertices)

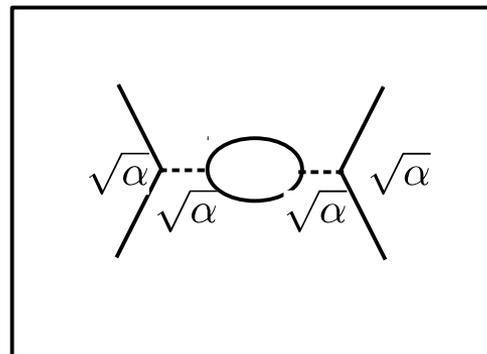
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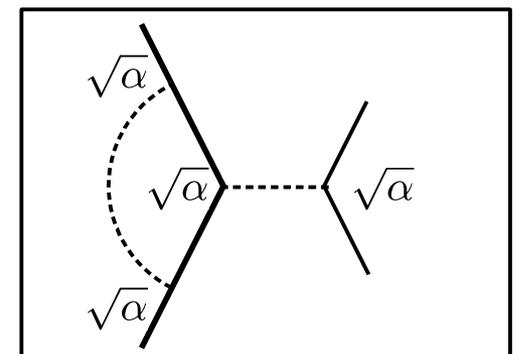
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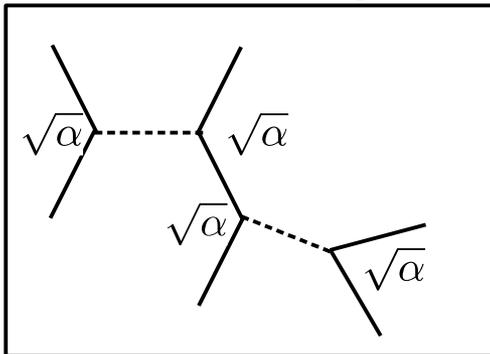
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- LO term for a $2 \rightarrow 4$ process.
- NLO contrib. for the $2 \rightarrow 2$ process.
- **Opens phasepace.**

Order α^2 diagrams (QED)

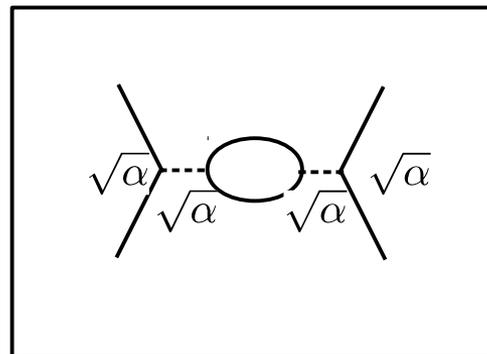
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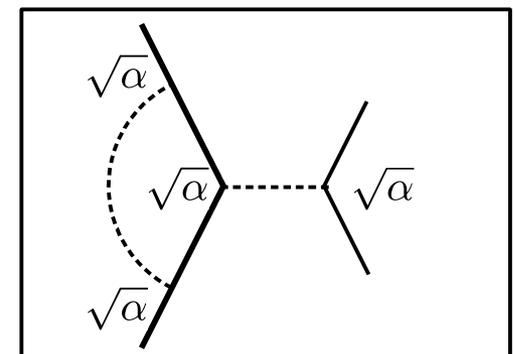
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Loops:



(in propagators or legs)

- Modifies (effective) masses of particles (**“running masses”**).

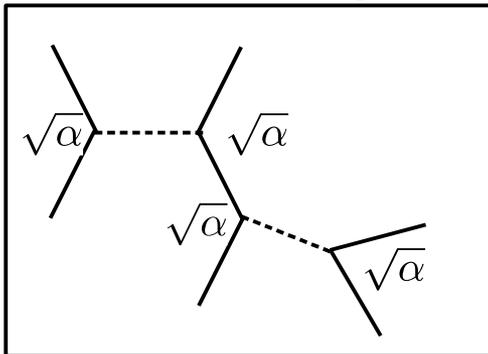


(in vertices)

Order α^2 diagrams (QED)

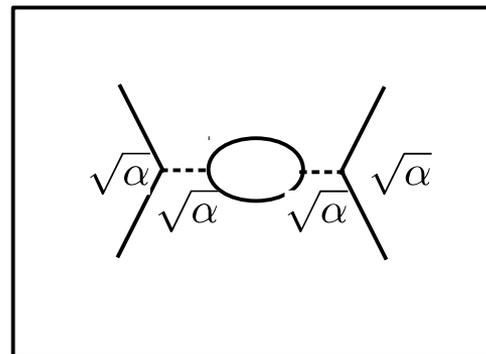
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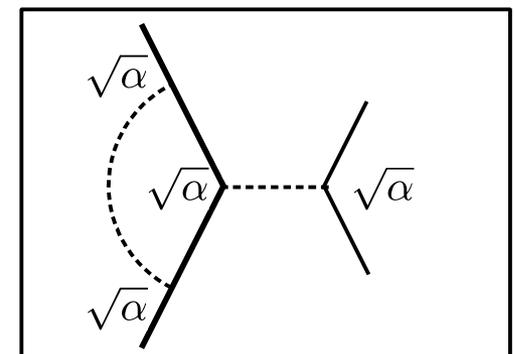
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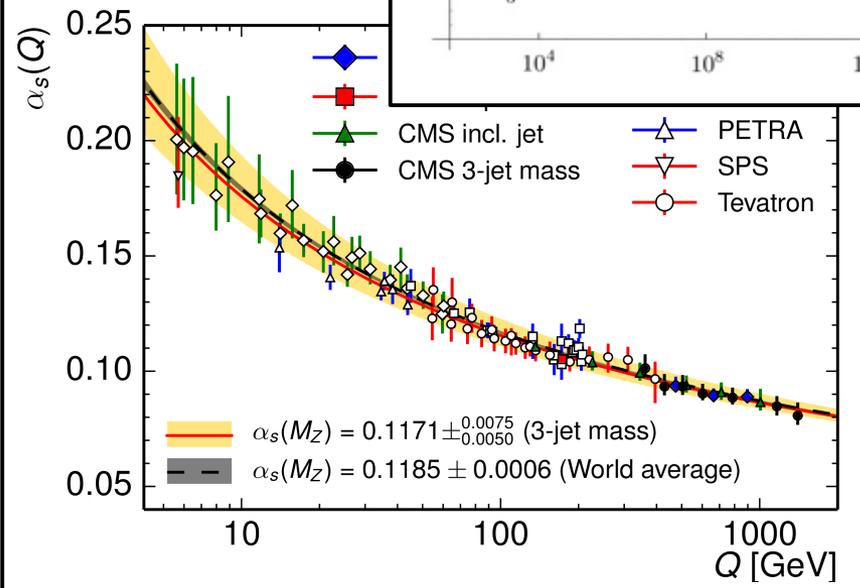
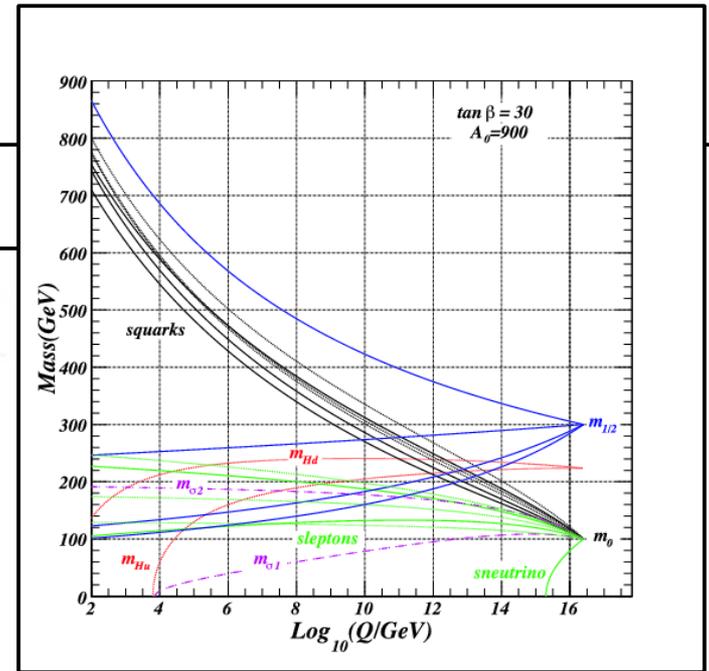
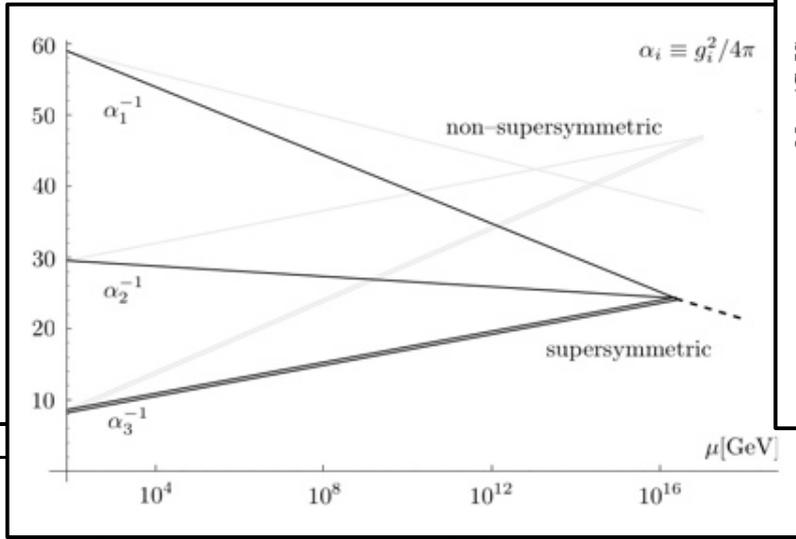
- Modifies (effective) masses of particles (“**running masses**”).



(in vertices)

- Modifies (effective) couplings of particles (“**running couplings**”).

Examples for “running constants”



- Running of the constants can be predicted and are indeed observed.
- Coupling needs to be measured at least in one point.
- One usually gives the value at a reference scale (e.g. m_Z).

Effect of higher order corrections

- **Change of over all normalization** of cross sections (e.g. via change of coupling, but also by kinematic opening of phasespace – large effect).
- **Change of kinematic distributions** (e.g. harder or softer transverse momentum spectrum of particles)

- **Change of over all normalization** of cross sections (e.g. via change of coupling, but also by kinematic opening of phasespace – large effect).
- **Change of kinematic distributions** (e.g. harder or softer transverse momentum spectrum of particles)
- **In QED effects are usually “small”** (correction to LO is already at $\mathcal{O}(1\%)$ level). **In QCD effects are usually “large”** ($\mathcal{O}(10\%)$). Therefore reliable QCD predictions almost always require (N)NLO calculations.
- Higher orders can be mixed (e.g. $\mathcal{O}(\alpha\alpha_s^2)$).
- In concrete calculations the number of contributing diagrams quickly explodes for higher order calculations, which makes these calculations very difficult.



Running of λ in the Higgs potential

- Like the couplings α , α_w and α_s also the **self-coupling λ in the Higgs potential is subject to higher order corrections:**

$$\mathcal{L}^{\text{Higgs}} = \partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi)$$

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \quad (\text{Higgs potential})$$

$$\frac{d\lambda}{d \log Q^2} = \frac{1}{16\pi^2} \left[\underbrace{12\lambda^2}_{\text{Higgs}} + \underbrace{6\lambda y_t^2 - 3y_t^4}_{\text{top quark}} - \frac{3}{2}\lambda (3\alpha^2 + \alpha_w^2) + \dots \right]$$

Higgs

top quark

(Renormalization group equation at 1-loop accuracy)

- Since the Higgs boson couples proportional to the mass the high energy behavior of λ will be dominated by the heaviest object in the loop.

Running of λ in the Higgs potential

- First case: large Higgs mass ($m_H \gg Q^2$).

$$\frac{d\lambda}{d \log Q^2} = \frac{1}{16\pi^2} \left[12\lambda^2 + 6\lambda y_t^2 - 3y_t^4 - \frac{3}{2}\lambda (3\alpha^2 + \alpha_w^2) + \dots \right]$$

\downarrow $m_H \gg Q^2$
Higgs top quark

$$\frac{d\lambda}{d \log Q^2} = \frac{3}{4\pi^2} \lambda^2(Q^2) \xrightarrow{\text{solution}} \lambda(Q^2) = \frac{\lambda(v^2)}{1 - \frac{3}{4\pi^2} \lambda(v^2) \log(Q^2/v^2)}$$

- For $Q^2 \ll v^2 = 246 \text{ GeV}$ we get $\log(Q^2/v^2) \rightarrow -\infty$ and $\lambda(Q^2) \rightarrow 0$.
- For increasing Q^2 $\lambda(Q^2)$ will run into a pole and become non-perturbative. This pole is called *Landau pole*. From the pole an upper bound on m_H can be obtained depending on the scale Q .

- The upper bound on m_H due to the *Landau* pole is called *triviality bound*:

$$m_H (Q = 10^3 \text{ GeV}) \leq 1.6 \text{ TeV}$$

$$m_H (Q = 10^{16} \text{ GeV}) \leq 340 \text{ GeV}$$

(triviality bound)

- NB: here Q indicates up to which scale the SM should be applicable.

The Running of λ in the Higgs Potential

- Second case: small Higgs mass ($m_H \ll m_t$)

$$\frac{d\lambda}{d \log Q^2} = \frac{1}{16\pi^2} \left[\underbrace{12\lambda^2 + 6\lambda y_t^2 - 3y_t^4}_{\text{Higgs}} - \underbrace{\frac{3}{2}\lambda(3\alpha^2 + \alpha_w^2)}_{\text{top quark}} + \dots \right]$$

$m_H \gg m_t$

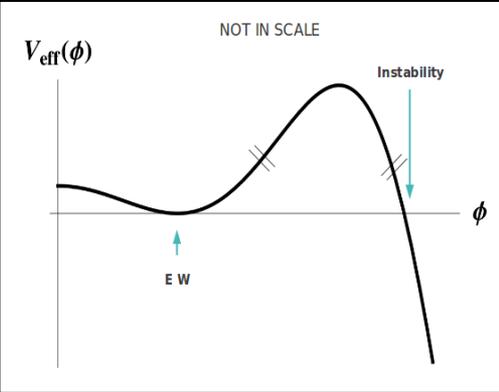
$$\frac{d\lambda}{d \log Q^2} = -\frac{3}{16\pi^2} y_t^4$$

solution \rightarrow

$$\lambda(Q^2) = \lambda(v^2) - \frac{3}{16\pi^2} \frac{m_t^4}{v^4} \log(Q^2/v^2)$$

(with: $y_t = m_t/v$)

- With increasing Q^2 $\lambda(Q^2)$ will turn negative and the Higgs potential will no longer be bound from below. The vacuum turns unstable. **From this turning point we obtain a lower bound on m_H depending on the scale Q .**



Higgs potential w/ running λ .

Triviality bound & stability bound

- The upper bound on m_H due to the *Landau* pole is called *triviality bound*:

$$m_H (Q = 10^3 \text{ GeV}) \leq 1.6 \text{ TeV}$$

$$m_H (Q = 10^{16} \text{ GeV}) \leq 340 \text{ GeV}$$

(triviality bound)

- The lower bound on m_H is called *stability bound*:

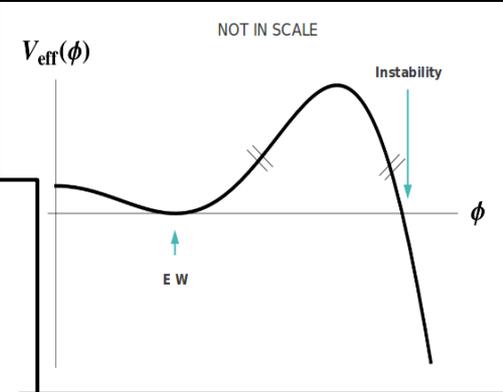
$$m_H (Q = 10^3 \text{ GeV}) \geq 20 \text{ GeV}$$

$$m_H (Q = 10^{16} \text{ GeV}) \geq 90 \text{ GeV}$$

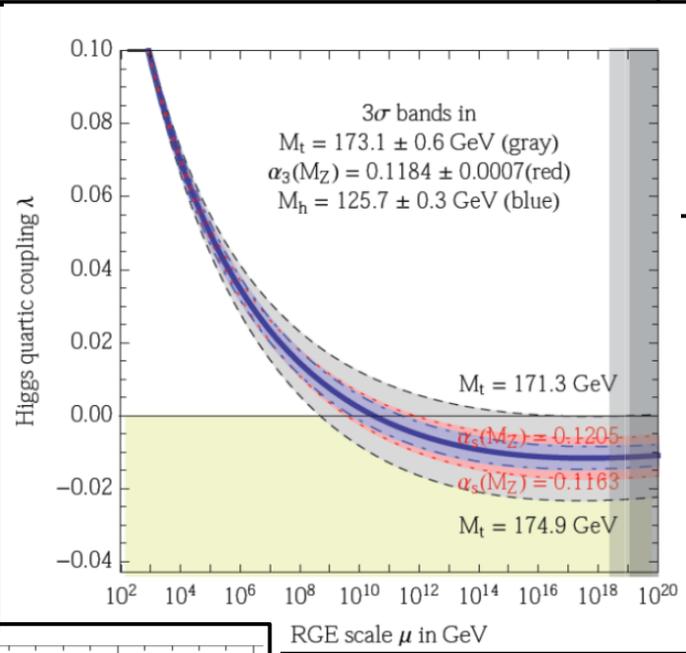
(stability bound)

- Indeed the later search window for the SM Higgs boson was in the range of $100 < m_H < 1000 \text{ GeV}$, for these and other reasons.
- NB: here Q indicates up to which scale the SM should be applicable.

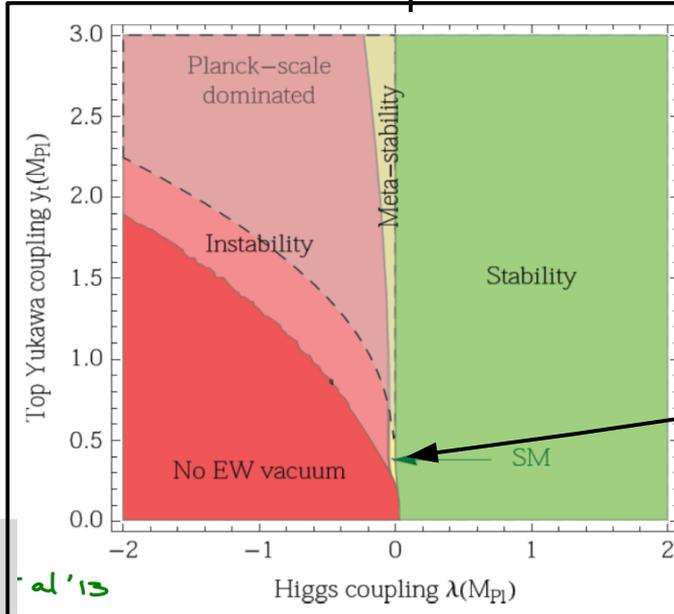
Intrinsic bounds on m_H



Higgs potential w/ running λ .



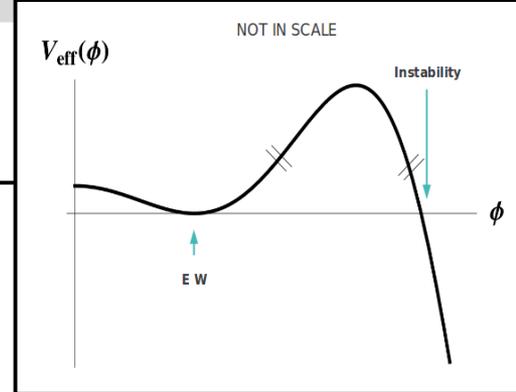
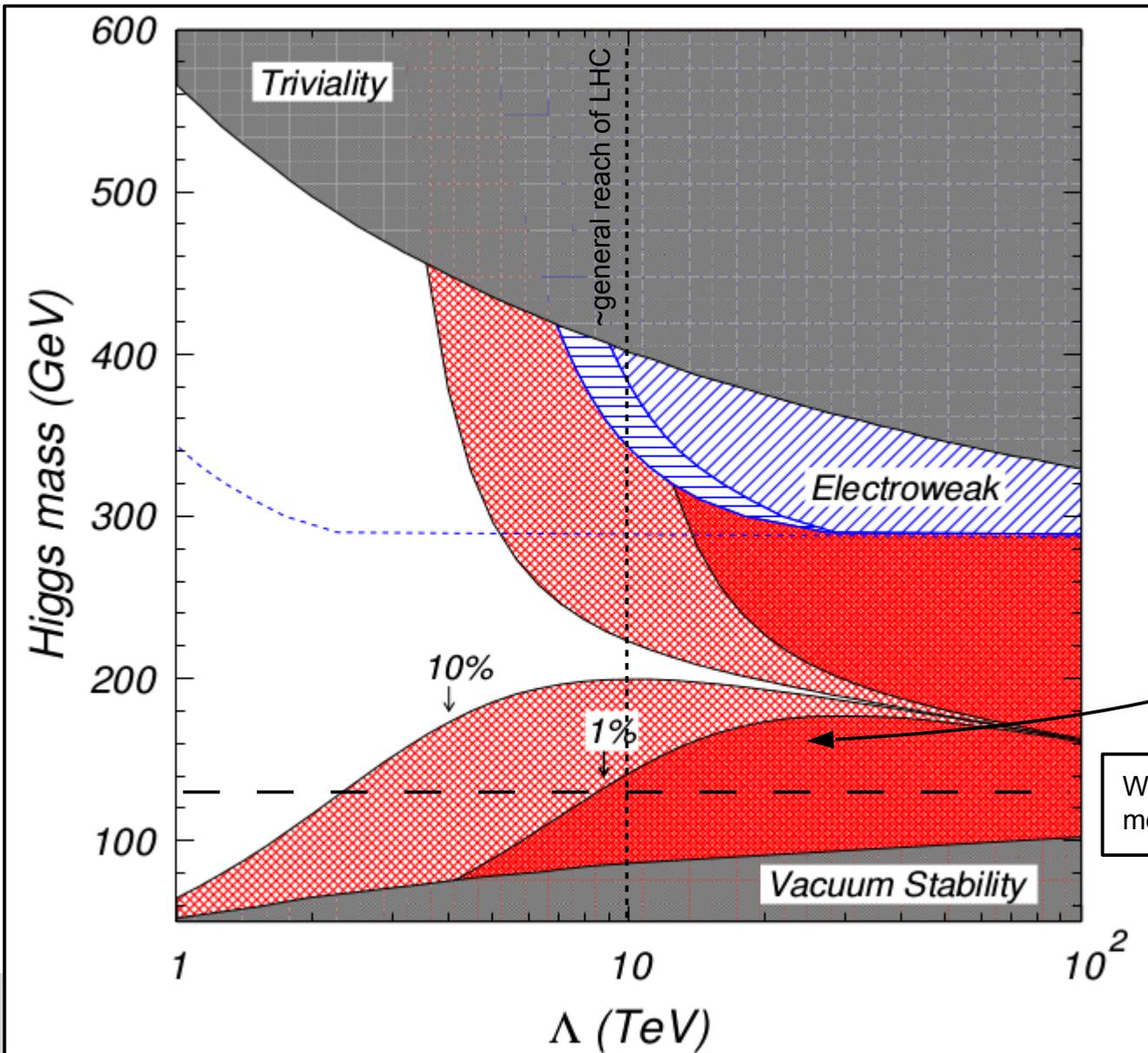
The SM in the stress field of vacuum stability.



Running of λ .



Intrinsic bounds on m_H



Higgs potential w/ running λ .

Different levels of fine tuning in the SM.

What we have found and measured for m_H .

Concluding Remarks

- Reviewed Feynman rules and calculated **cross section for simple QED scattering** process.
- Briefly discussed effects of **higher order corrections** in perturbation theory.
- Discussed **boundaries on Higgs boson mass** immanent to the SM as an application of higher order effects on the Higgs self-coupling.
- Note: on Thursday next week will be holiday. On Friday next week there will be an Exercise session. The week after we will start with the experimental part of the lecture.

