Karlsruhe Institute of Technology

Moderne Experimentalphysik III: Kerne und Teilchen (Physik VI)

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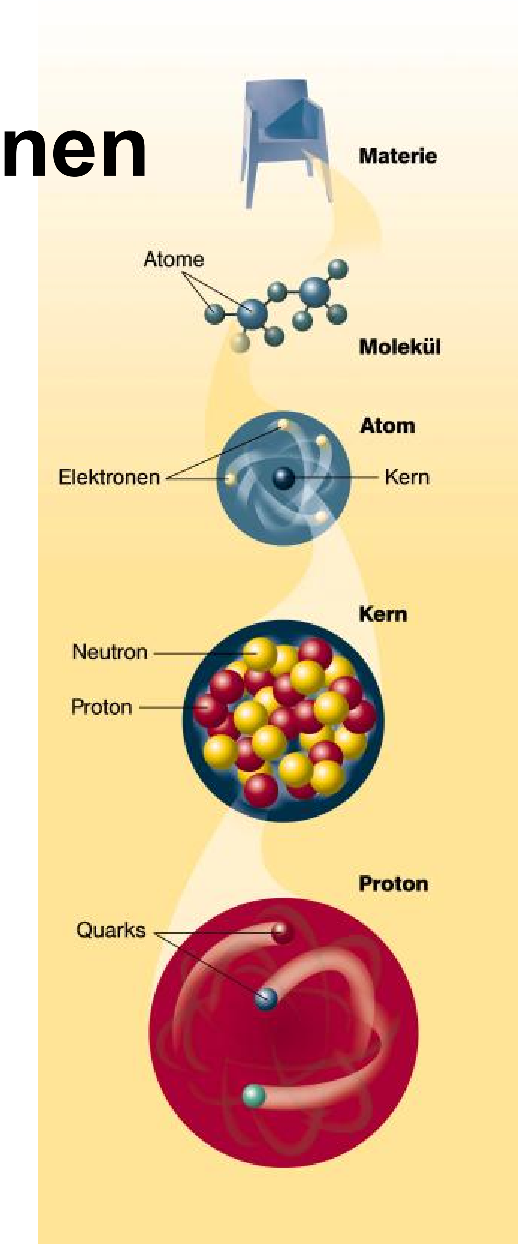
23. Mai 2017

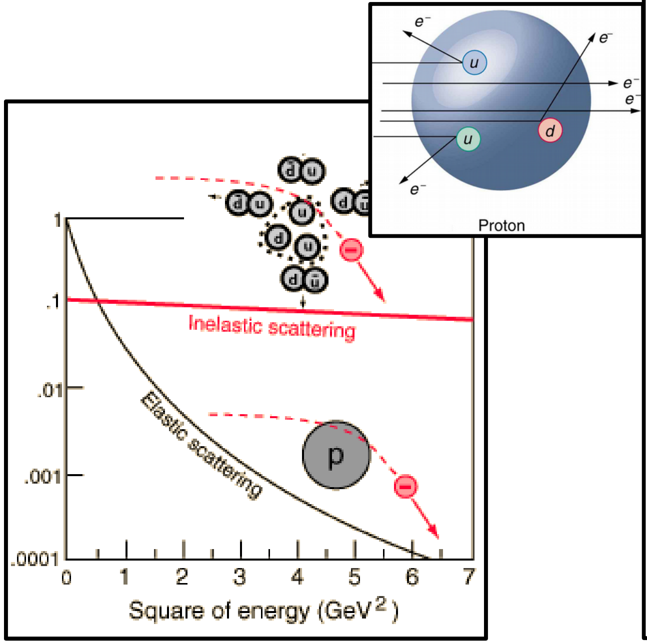
INSTITUTE OF EXPERIMENTAL PARTICLE PHYSICS (IEKP) – PHYSICS FACULTY

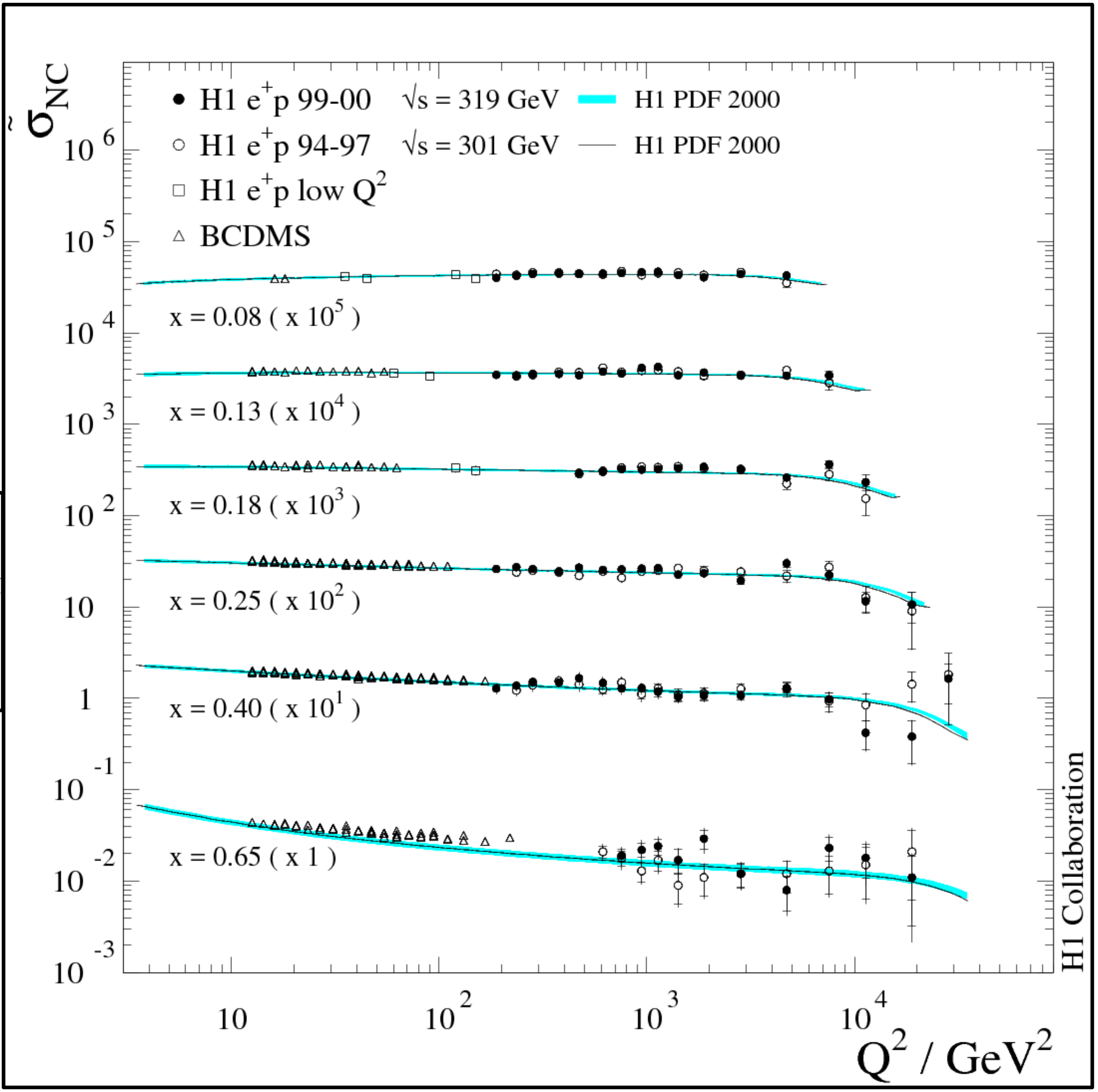
Kapitel 3.2: Struktur der Nukleonen



Klärung der Proton-Substruktur

- $\tilde{\sigma}_{\mathrm{NC}}$: **normiert auf Dirac-Wirkungsquerschnitt** (→ punktförmiges Spin-½ Teilchen)
- Skalenverhalten
- Proton besteht aus **punktförmigen Konstituenten**





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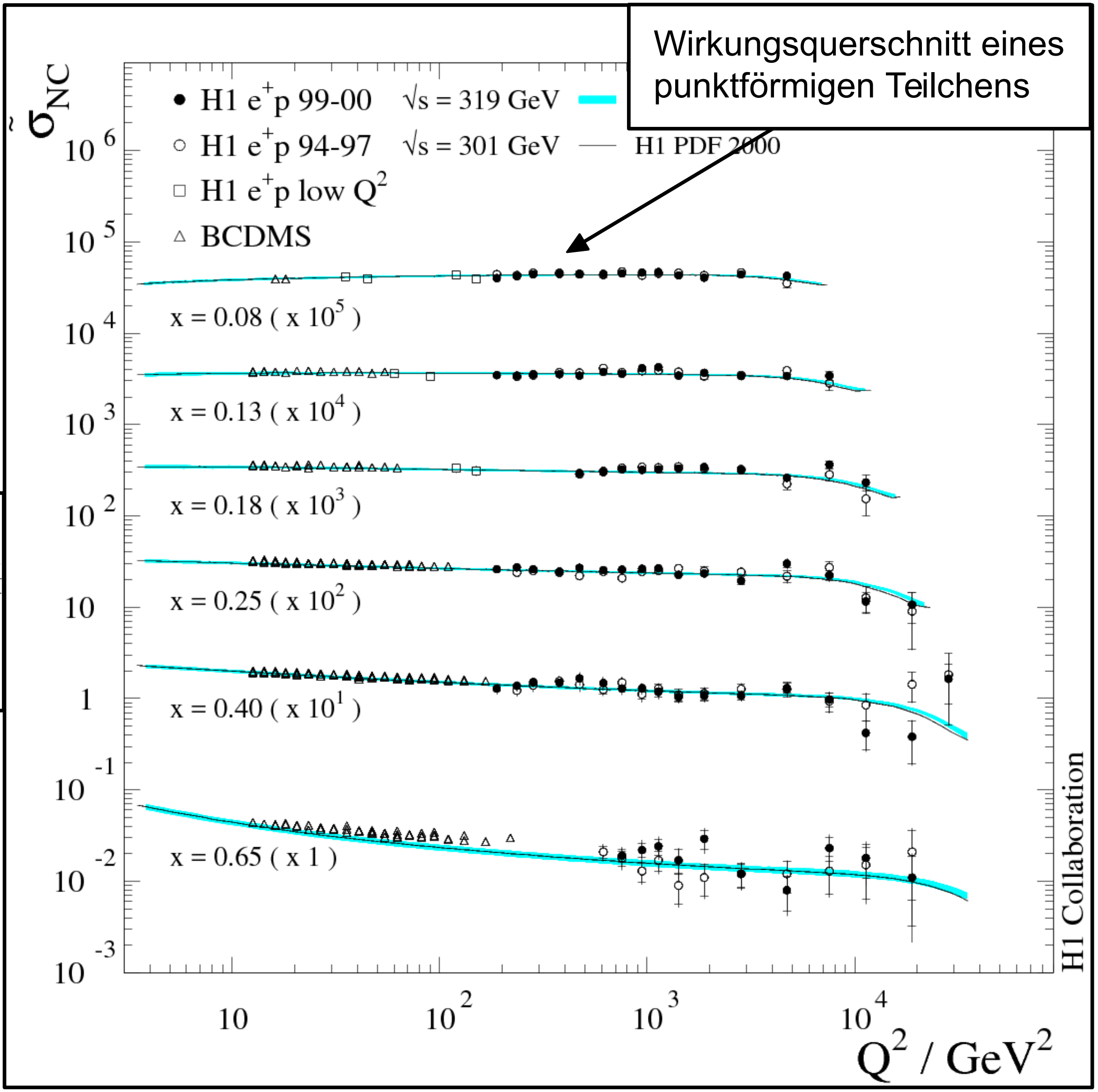
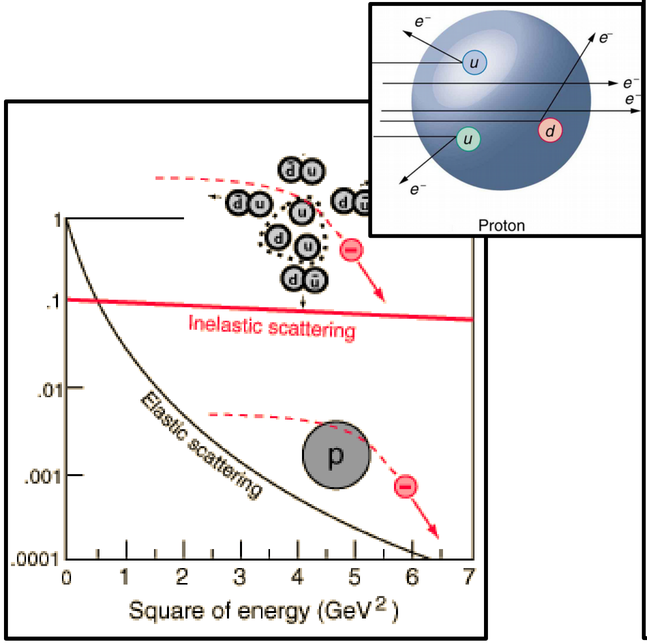

Wirkungsquerschnitt eines punktförmigen Teilchens



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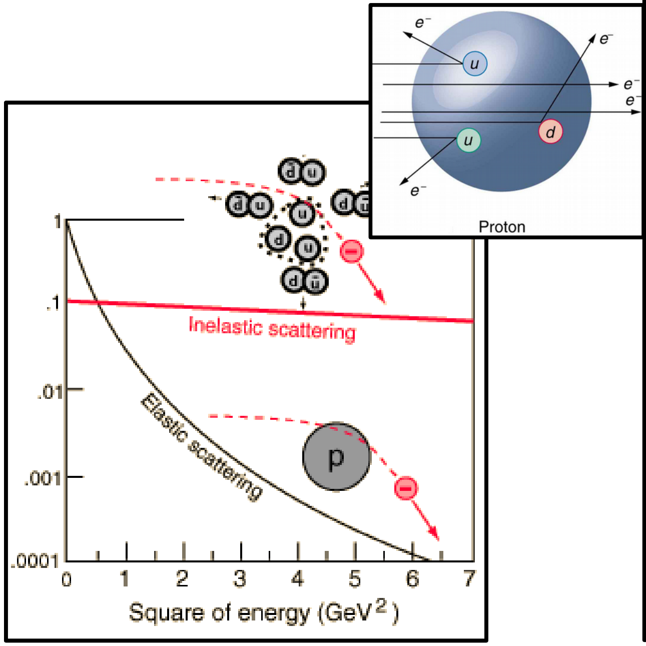
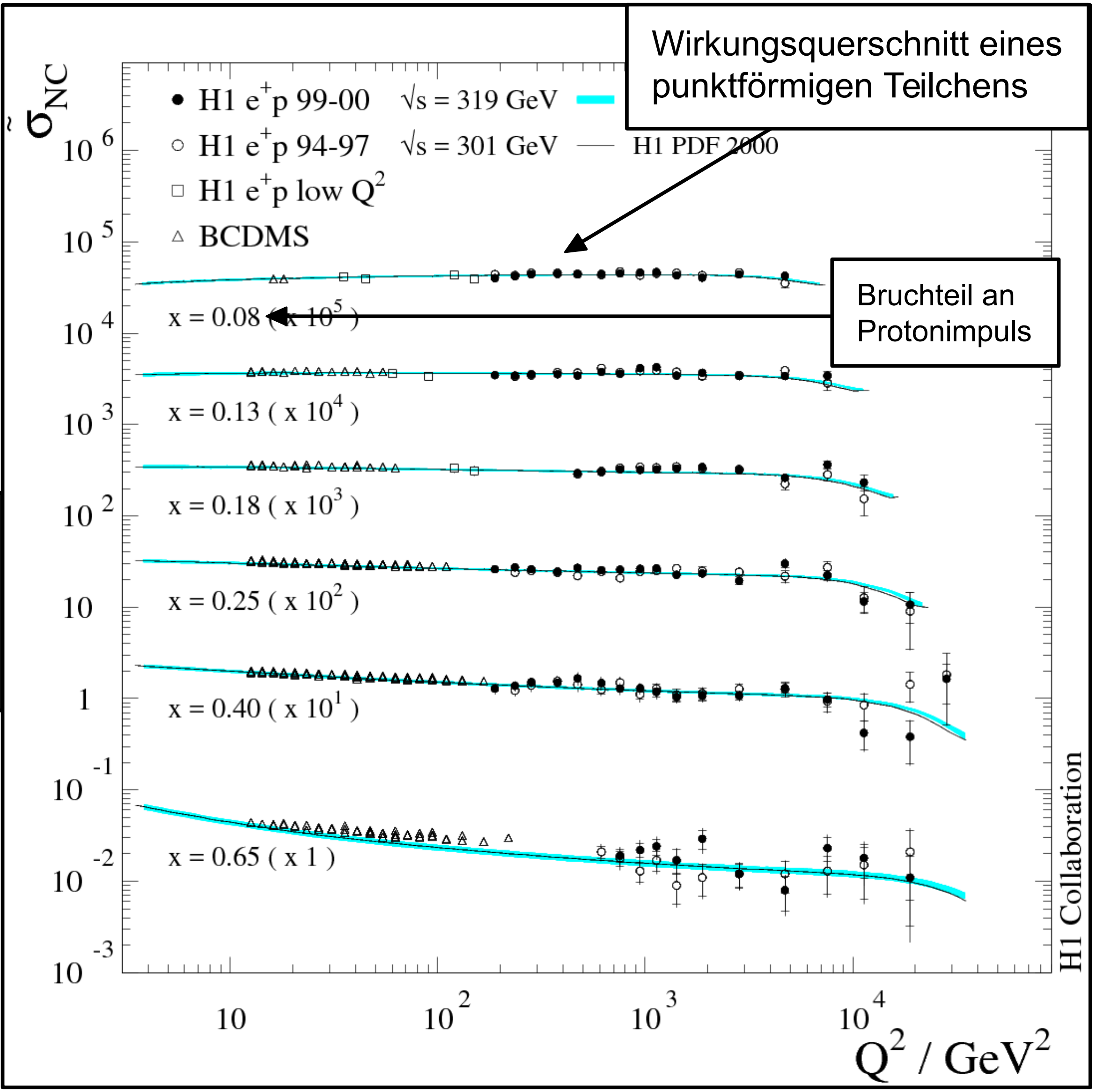
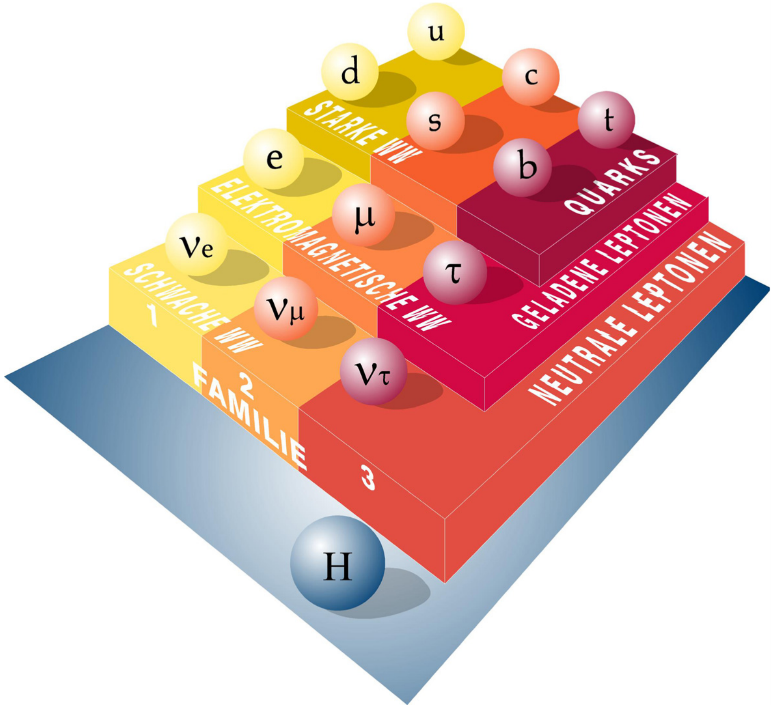




Wirkungsquerschnitt eines punktförmigen Teilchens

Bruchteil an Protonimpuls

Kapitel 3.3: Fundamentaler Aufbau der Materie und ihre Wechselwirkungen



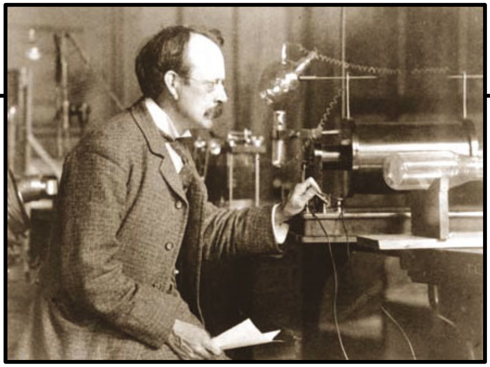
Woraus die Welt wirklich besteht…

120 Jahre Physik auf der Suche nach den letzten Bausteinen der Materie

Bestandsaufnahme:

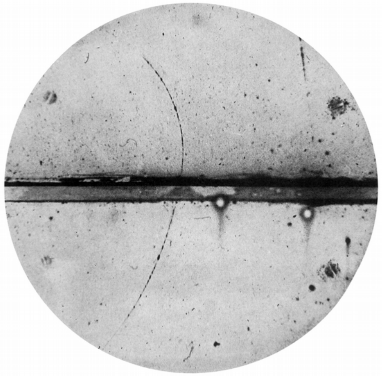
- Elektron → **punktförmig**, besitzt schwere Brüder
- Atomkerne → bestehen aus Nukleonen
- Nukleonen → bestehen Partonen (Quarks)
- Partonen → **punktförmig**

Discovery of the electron (1897)

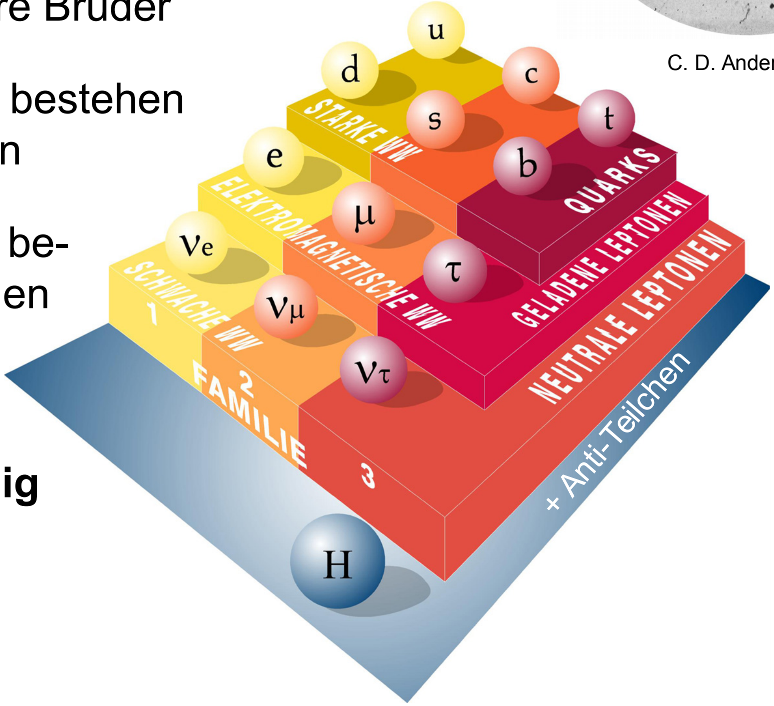


J. J. Thomson (1856 – 1940)

Discovery of the positron (1932)



C. D. Anderson (1905 – 1991)



- **Spin-½ Fermionen**

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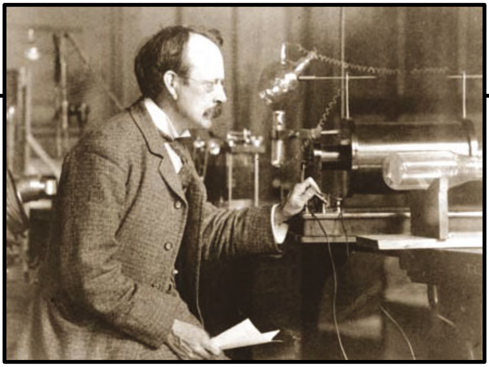

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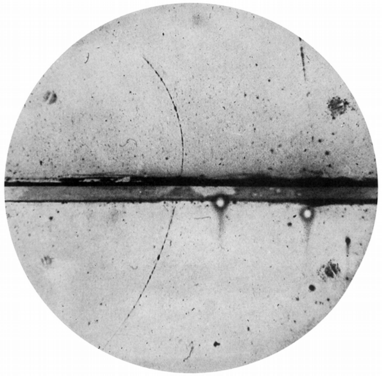

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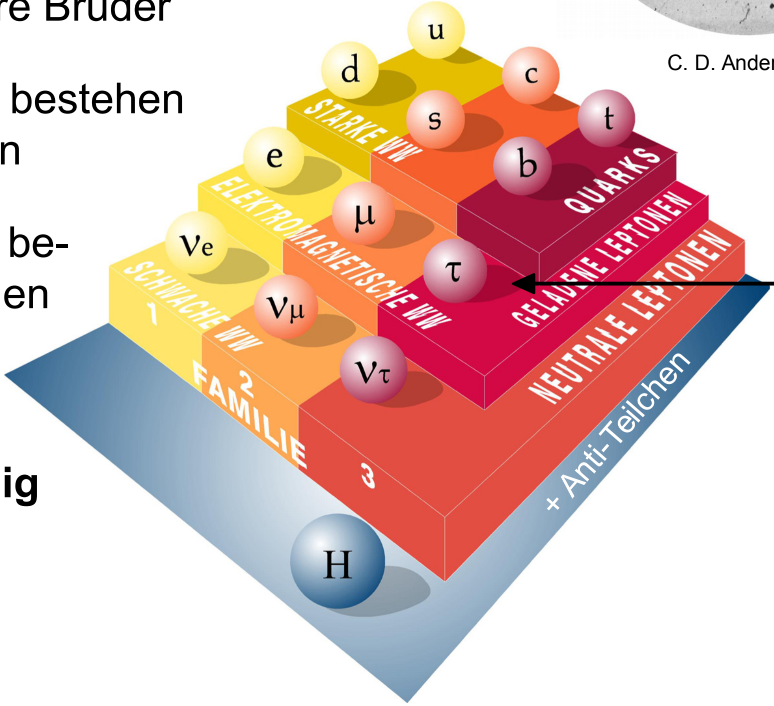

Nach unserer bisherigen gesicherten Erkenntnis ist **das hier** der Stoff aus dem die Welt die uns umgibt besteht

Was die Welt zusammenhält…

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Bestandsaufnahme:

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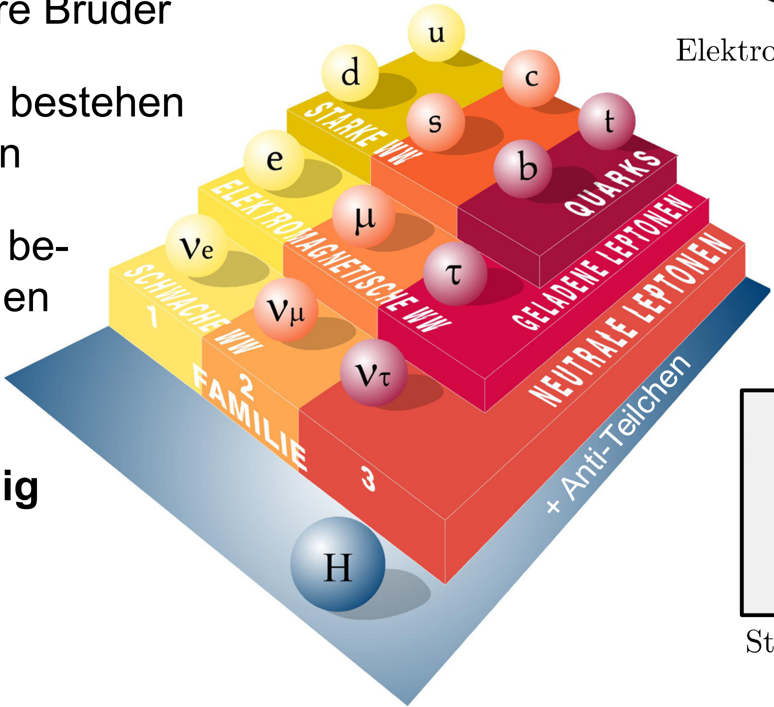
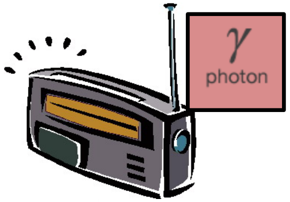


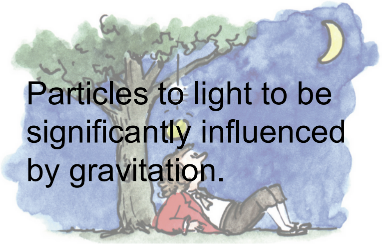
- **Spin-½ Fermionen**

Kräfte/Wechselwirkungen:

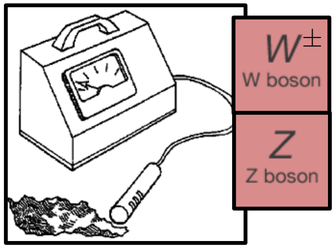
Wir kennen **vier fundamentale Kräfte** im Universum:



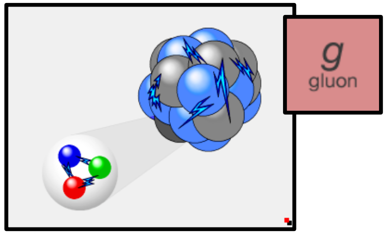
Elektromagnetismus



Particles to light to be significantly influenced by gravitation.



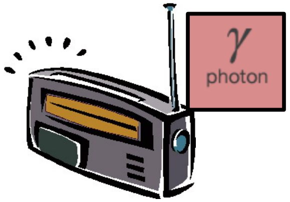
Schwache Kraft



Starke Kraft

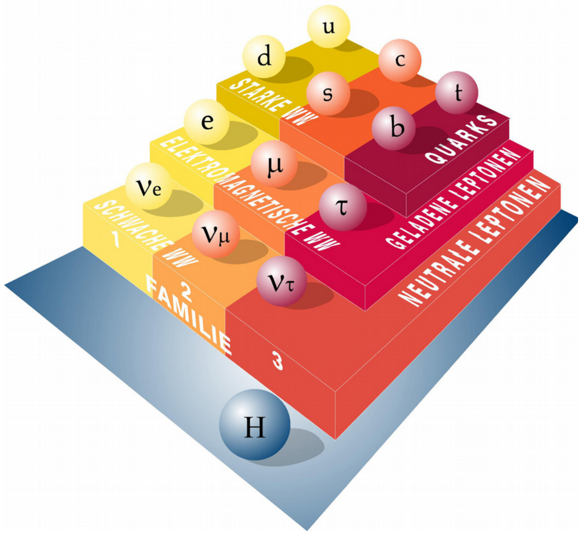
- **Bosonen mit Spin-1**

Fundamentale Wechselwirkungen



Elektromagnetismus

- Koppelt an **elektrische Ladung**
- Kann abstoßend oder anziehend wirken



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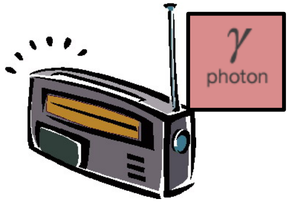

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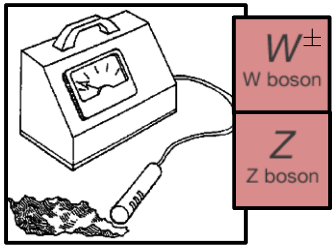
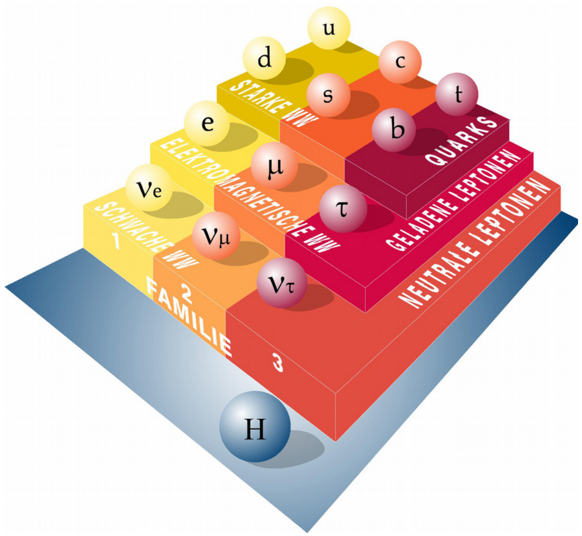


Schwache Kraft

- Koppelt an **schwachen Isopsin**
- Kann geladene Teilchen in ungeladene Teilchen umwandeln
- Verantwortlich für die folgenden Reaktionen:

$n \to p\, e^- \, \bar{\nu}_e$ (Neutronzerfall)

$\nu_e e^- \to \nu_e \, e^-$ (elastische $e\,\nu$-Streuung)



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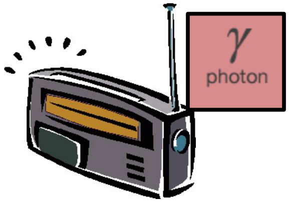

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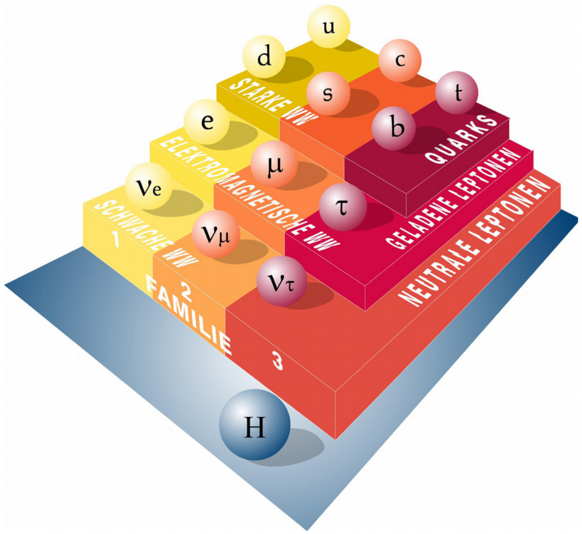

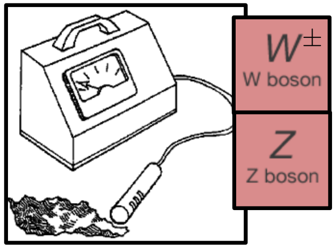

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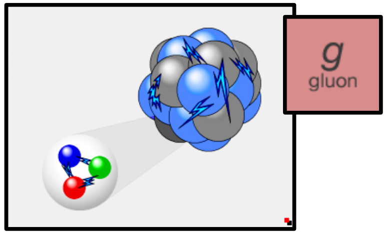
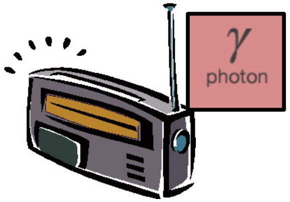


Starke Kraft

- Koppelt an **Farbladung** (→ rot, grün, blau)
- Wirkt auf Entfernungen eines Kerns stärker als em WW
- Fällt jenseits dieser Entfernung sofort auf Null ab (→ Kastenpotential)

Fundamentale Wechselwirkungen



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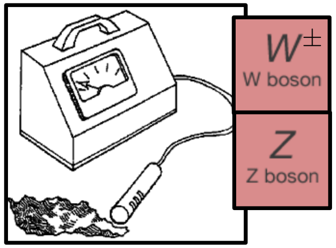

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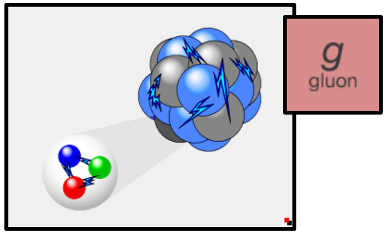


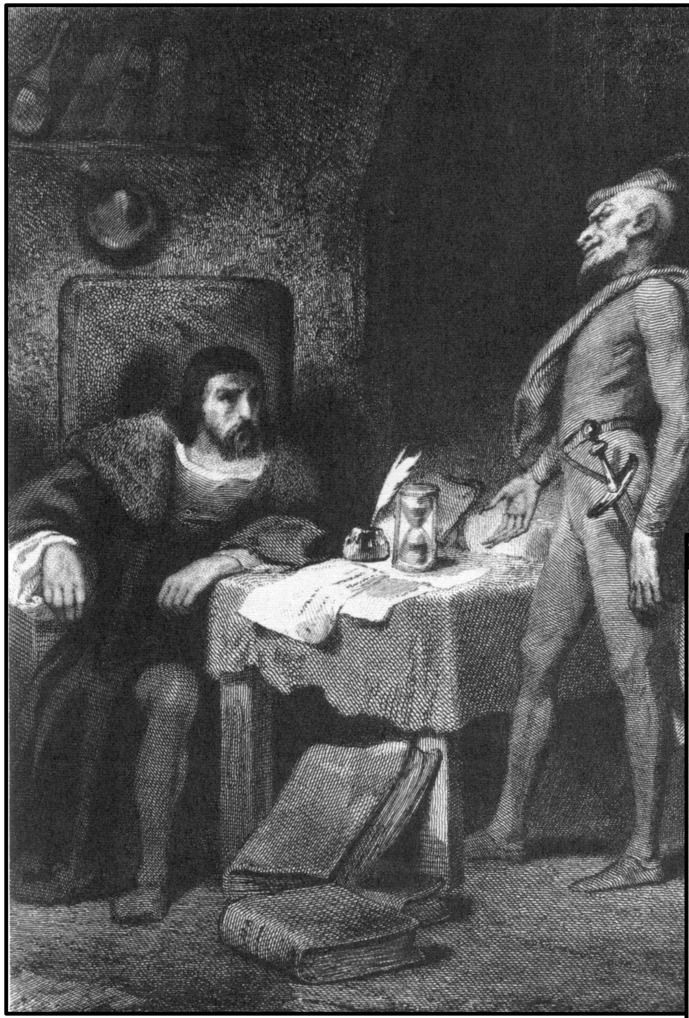
Starke Kraft

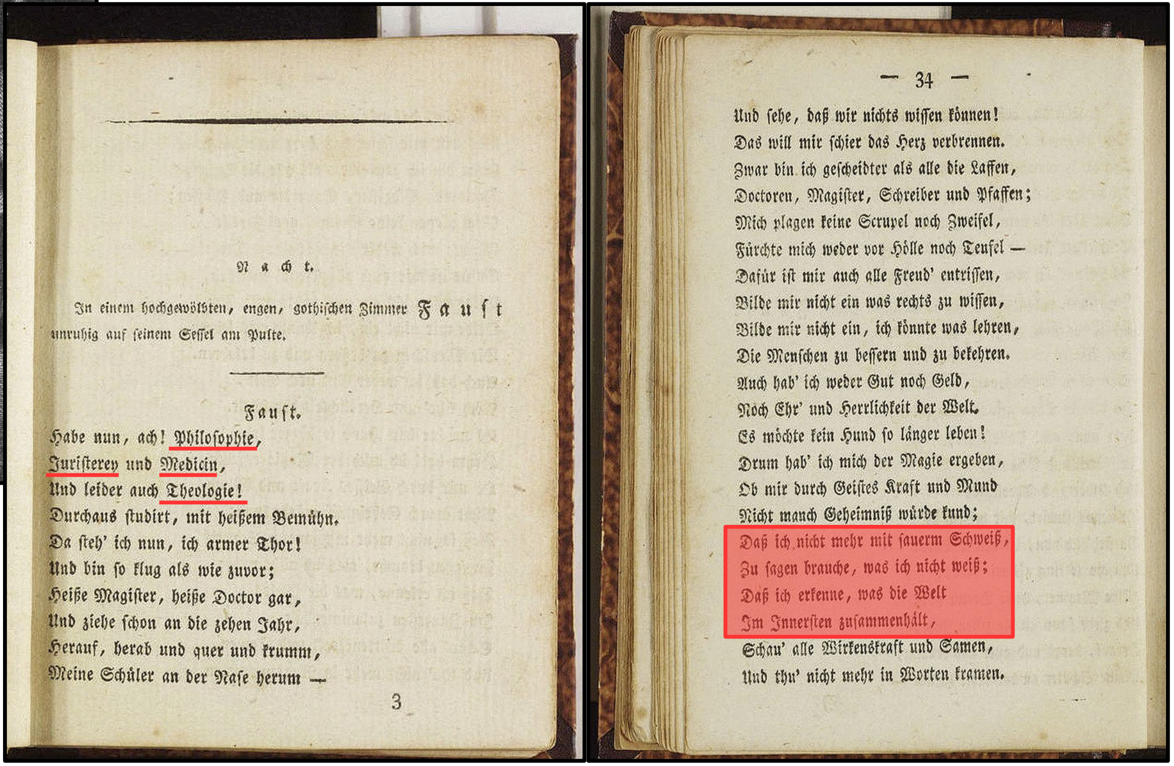
Wechselwirkung	starke WW	schwache WW	em WW	Graviation
Kopplung (α)	$\mathcal{O}(1)$	$\mathcal{O}(10^{-3})$	$\mathcal{O}(1/137)$	$\mathcal{O}(10^{-39})$
Austauschteilchen	Gluonen	W, Z	γ	Graviton
Masse Austauscht.	$< \mathcal{O}(\text{MeV})^{(1)}$	90 GeV	0	$< 1.2 \cdot 10^{-22}\text{eV}^{(1)}$
Rel. Stärke $^{(2)}$	1	10^{-13}	10^{-3}	10^{-38}
Reichweite [m]	10^{-15}	10^{-15}	∞	∞
typische Zeitskala [s]	10^{-23}	10^{-10}	10^{-20}	?

(1) theoretisch 0 (2) im Abstand von 1 fm

Theoretische Beschreibung

- Noble Ziele…
- Etwa 220 Jahre später
- Die gleichen Fragen…





Nacht.

In einem hochgewölbten, engen, gothiſchen Zimmer **Fauſt** unruhig auf ſeinem Seſſel am Pulte.

Fauſt.

Habe nun, ach! Philoſophie,
Juriſterey und Medicin,
Und leider auch Theologie!
Durchaus ſtudirt, mit heißem Bemühn.
Da ſteh' ich nun, ich armer Thor!
Und bin ſo klug als wie zuvor;
Heiße Magiſter, heiße Doctor gar,
Und ziehe ſchon an die zehen Jahr,
Herauf, herab und quer und krumm,
Meine Schüler an der Naſe herum —

3

— 34 —

Und ſehe, daß wir nichts wiſſen können!
Das will mir ſchier das Herz verbrennen.
Zwar bin ich geſcheidter als alle die Laffen,
Doctoren, Magiſter, Schreiber und Pfaffen;
Mich plagen keine Scrupel noch Zweifel,
Fürchte mich weder vor Hölle noch Teufel —
Dafür iſt mir auch alle Freud' entriſſen,
Bilde mir nicht ein was rechts zu wiſſen,
Bilde mir nicht ein, ich könnte was lehren,
Die Menſchen zu beſſern und zu bekehren.
Auch hab' ich weder Gut noch Geld,
Noch Ehr' und Herrlichkeit der Welt.
Es möchte kein Hund ſo länger leben!
Drum hab' ich mich der Magie ergeben,
Ob mir durch Geiſtes Kraft und Mund
Nicht manch Geheimniß würde kund;
Daß ich nicht mehr mit ſauerm Schweiß,
Zu ſagen brauche, was ich nicht weiß;
Daß ich erkenne, was die Welt
Im Innerſten zuſammenhält,
Schau' alle Wirkenskraft und Samen,
Und thu' nicht mehr in Worten kramen.

J. W. von Goethe, Der Tragödie erster Teil, Tübingen: Cotta, 1808

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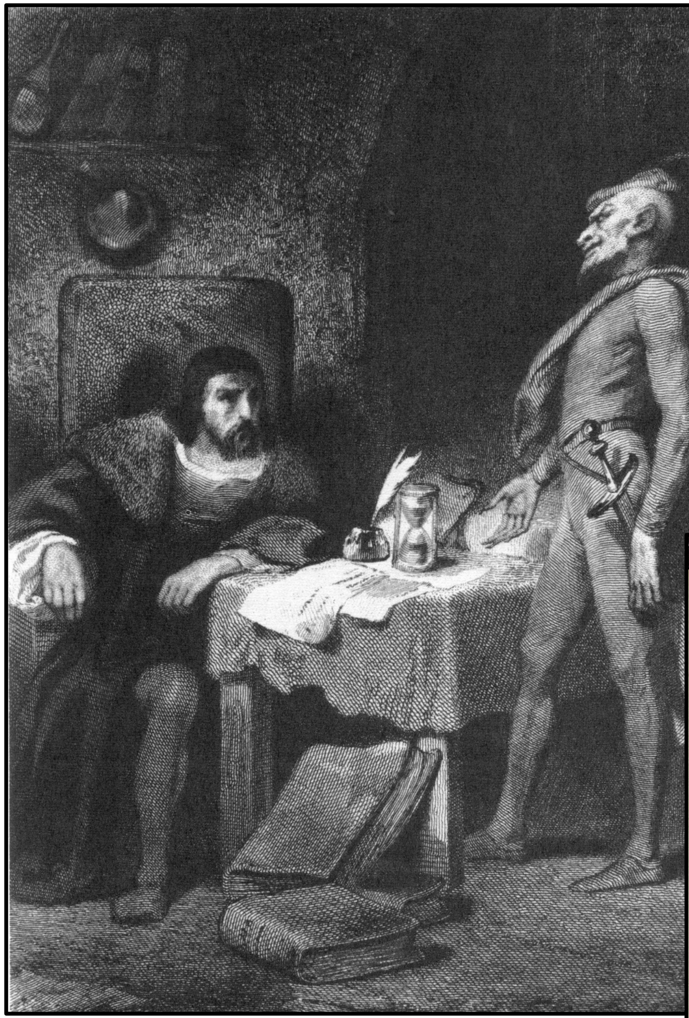
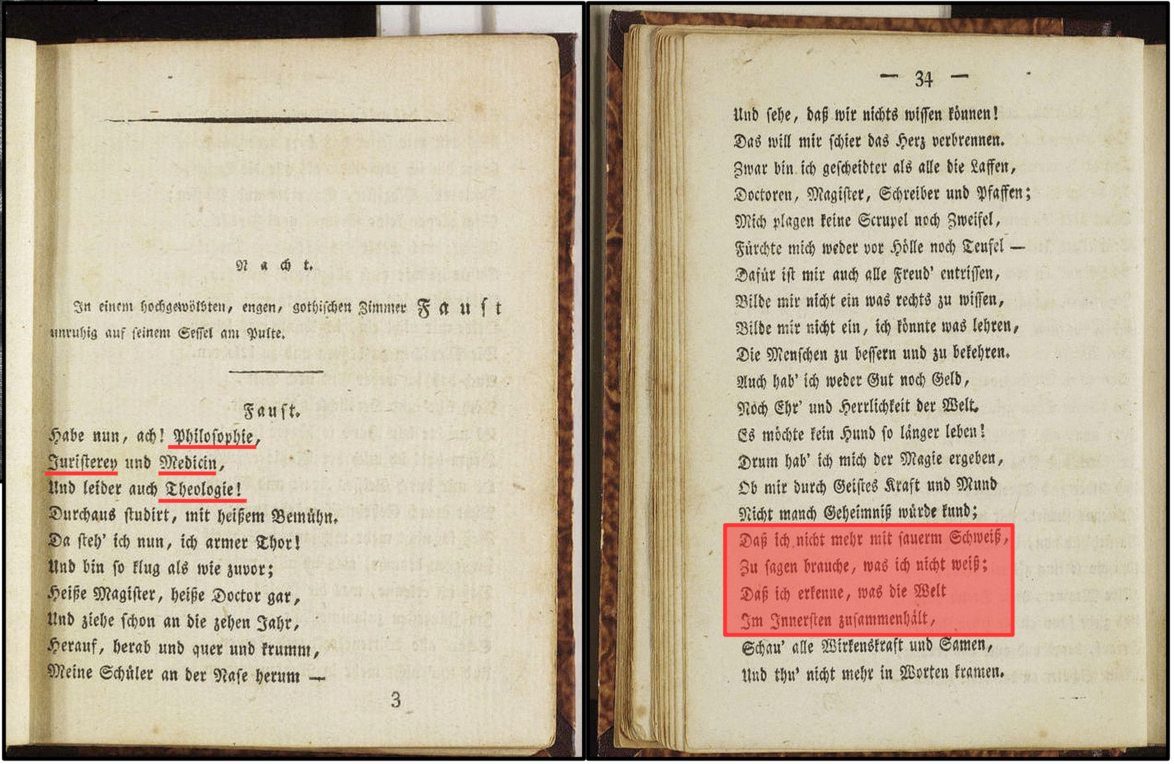


… etwas erfolgsorientiertere Lösungsansätze



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Drei Säulen des Standardmodells

Quantenfeldtheorie

- Relativistiche QM.
- Erzeugung/Vernichtung von Teilchen.

Symmetrien

- Fundamentale WW.
- Struktur der Materie

Symmetriebrechung

- Teilchenmasse (hier noch nicht diskutiert).

Klein-Gordon/Dirac-Gleichung

- Beschreibung **relativistischer Prozesse** der Teilchenphysik:

$$E^2 - p^2 = m^2$$

Klein-Gordon/Dirac-Gleichung

- Beschreibung **relativistischer Prozesse** der Teilchenphysik:

$$E^2 - p^2 = m^2 \quad \Longrightarrow \quad \boxed{\begin{aligned} E &\rightarrow \quad i\partial_t \\ \vec{p} &\rightarrow -i\vec{\nabla} \end{aligned}}$$

Kanonische Ersetzung

Klein-Gordon/Dirac-Gleichung

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Kanonische Ersetzung

Dynamik von **Bosonen**

$$\boxed{\left(\partial_\mu \partial^\mu + m^2\right)\phi = 0}$$

(Klein-Gordon-GL)

Klein-Gordon/Dirac-Gleichung

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Dynamik von **Bosonen**

$$E^2 - p^2 = m^2 \quad \Rightarrow \quad \boxed{\begin{array}{l} E \to \quad i\partial_t \\ \vec{p} \to -i\vec{\nabla} \end{array}} \quad \Rightarrow \quad \boxed{\left(\partial_\mu \partial^\mu + m^2\right)\phi = 0}$$

Kanonische Ersetzung

(Klein-Gordon-GL)

- Lösungen:

$$\phi_+(\vec{x},t) = u(\vec{p})e^{+i(\vec{p}\vec{x}-Et)}$$

$$\phi_-(\vec{x},t) = v(\vec{p})e^{-i(\vec{p}\vec{x}-Et)} \qquad E(\vec{p}) = \sqrt{m^2 + \vec{p}^2}$$

(Ebene Welle)

- Besonderheit: Hamilton-Operator **nicht-lokal**:

$$\hat{H}_0 = \sqrt{m^2 - \vec{\nabla}^2} = m\sqrt{1 - \frac{\vec{\nabla}^2}{m^2}} = m - \frac{\vec{\nabla}^2}{2m} + \cdots$$ (*)

Klein-Gordon/Dirac-Gleichung

- Beschreibung **relativistischer Prozesse** der Teilchenphysik:

Dynamik von **Bosonen**

$$\left(\partial_\mu \partial^\mu + m^2\right) \phi = 0$$

(Klein-Gordon-GL)

- Linearisierte Form:

Dynamik von **Fermionen**

$$i\partial_t \psi = \hat{H}_0 \psi = \left(-i\vec{\alpha}\vec{\nabla} + \beta m\right) \psi$$

(Dirac-GL)

$\vec{\alpha}$ und β sind keine einfachen Zahlen (*) sondern algebraische Operatoren. Lassen sich als Matrizen darstellen.

Klein-Gordon/Dirac-Gleichung

- Beschreibung **relativistischer Prozesse** der Teilchenphysik:

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Bosonen

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Zweifache Anwendung muß auf Klein-Gordon-GL zurückführen

$$\left(i\partial_t\right)^2\psi = \left(-i\vec{\alpha}\vec{\nabla} + \beta m\right)^2\psi$$

$$= \Big[-\underbrace{\left(\alpha_i\alpha_j + \alpha_j\alpha_i\right)}_{\{\alpha_i,\alpha_j\} = 2\delta_{ij}}\partial_i\partial_j{}_{(i\leq j)} - im\underbrace{\left(\alpha_i\beta + \beta\alpha_i\right)}_{\{\alpha_i,\beta\} = 0}\partial_i + \underbrace{\left(\beta m\right)^2}_{\beta^2 = 1}\Big]\psi \overset{!}{\equiv} \Big[-\vec{\nabla}^2 + m\Big]\psi$$

(Antikommutator-Relationen)

Dirac Darstellung

- Konkrete Darstellung der Matrizen α_i und β :

$$\beta \;=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \alpha_i = \begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix}$$

($\sigma_i(i = 1, 2, 3)$ Pauli-Matrizen)

$$\sigma_1 \;=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$\sigma_2 \;=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

$$\sigma_3 \;=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Dirac Darstellung

- Manifest relativistisch-kovariante Formulierung mit Hilfe der $\gamma^{\mu\,(1)}$-Matrizen:

$$\gamma^0 \equiv \beta \qquad\qquad \gamma^i \equiv \beta\alpha_i$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad\qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$$\boxed{\begin{matrix} \{\alpha_i, \alpha_j\} = 2\delta_{ij} \\ \{\alpha_i, \beta\} = 0 \\ [\beta, \beta] = 0 \end{matrix} \quad \Longrightarrow \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}}$$

(Kompakte Schreibweise der Algebra)

- Dirac-Gleichung in relativistisch kovarianter Form:

$$\boxed{(i\gamma^\mu\partial_\mu - m)\,\psi = 0}$$

(Dirac-Gleichung)

Muß mindestens 4-dim haben, sonst lassen sich Kommutator-Relationen nicht erfüllen (2)

---

(1) Formelles Transformationsverhalten eines Lorentzvector.

(2) siehe Backup.

Lösungen der Dirac-Gleichung

$$\boxed{(i\gamma^\mu\partial_\mu - m)\,\psi = 0}$$

$$\psi_+(\vec{x}) = u(\vec{p})e^{+i(\vec{p}\vec{x}-Et)}$$

$$\psi_-(\vec{x}) = v(\vec{p})e^{-i(\vec{p}\vec{x}-Et)} \qquad E(\vec{p}) = \sqrt{m^2+\vec{p}^2}$$

(Ebene Welle)

$$\Big\downarrow \text{Spinoren}$$

$$\begin{array}{rc} \text{for } +m & u_\uparrow(0) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad u_\downarrow(0) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \\ \text{for } -m & v_\uparrow(0) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad v_\downarrow(0) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \end{array}$$

at rest ($\vec{p} \equiv 0$)

Lösungen der Dirac-Gleichung

$$\boxed{\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi = 0}$$

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(Ebene Welle)

Spinoren

(Lorentz Transformation)

$\Lambda : (m,0,0,0) \rightarrow (E, p_x, p_y, p_z)$

for $+m$

$$u_{\uparrow}(0) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad u_{\downarrow}(0) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

for $-m$

$$v_{\uparrow}(0) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad v_{\downarrow}(0) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

at rest ($\vec{p} \equiv 0$)

for $+m$

$$u_{\uparrow}(\vec{p}) = N\begin{pmatrix}E+m\\0\\p_z\\p_x+ip_y\end{pmatrix} \qquad u_{\downarrow}(\vec{p}) = N\begin{pmatrix}0\\E+m\\p_x-ip_y\\-p_z\end{pmatrix}$$

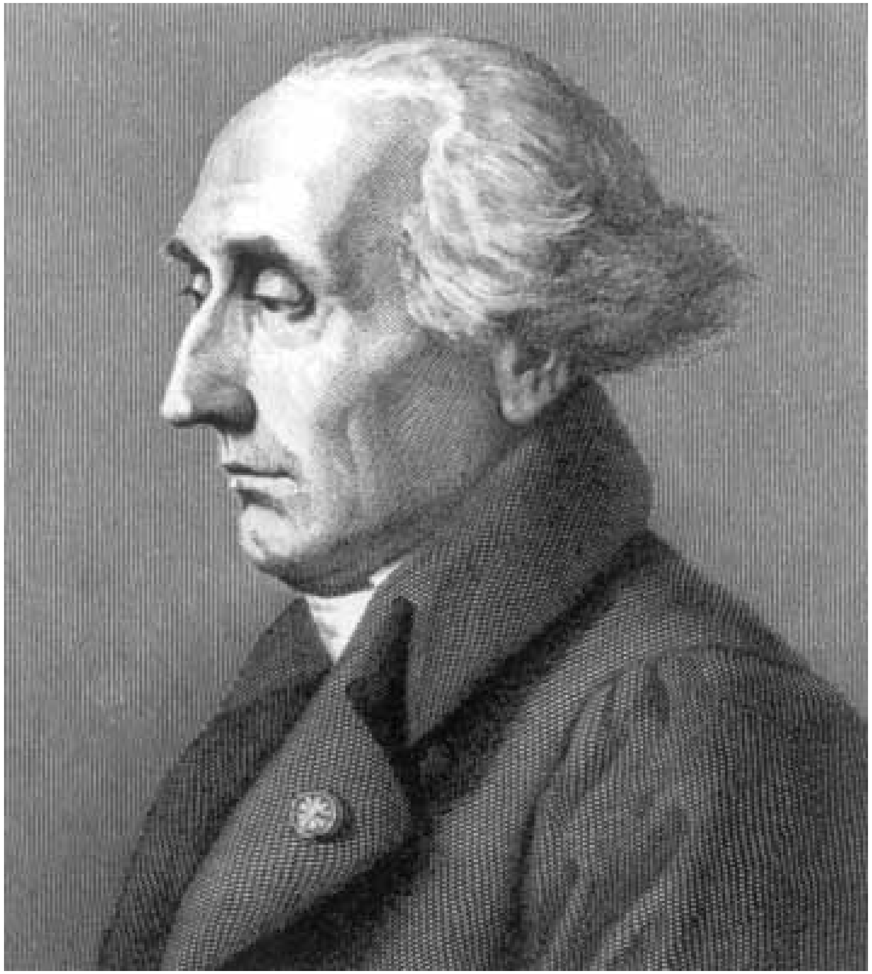
for $-m$

$$v_{\uparrow}(\vec{p}) = N\begin{pmatrix}p_z\\p_x+ip_y\\E+m\\0\end{pmatrix} \qquad v_{\downarrow}(\vec{p}) = N\begin{pmatrix}p_x-ip_y\\-p_z\\0\\E+m\end{pmatrix}$$

$$N = \frac{1}{\sqrt{2m(E+m)}}$$

in motion ($\vec{p} \neq 0$)

Lagrange Formalismus & Eichtransformationen



Joseph-Louis Lagrange
(*25. January 1736, † 10. April 1813)

Lagrange Formalismus (klassische Feldtheorie)

- Alle Informationen eines physikalischen Systems in **Wirkungsintegral** kodiert:

Wirkung

$$\boxed{\mathcal{S} = \int \mathcal{L}(\partial_\mu \phi, \phi)\mathrm{d}^4 x_\mu}$$

Feld ϕ (Verallgemeinerung generalisierte Koordinaten)

Lagrangedichte $\mathcal{L} \propto$

$$\left(\sum E_{\mathrm{kin}} - \sum E_{\mathrm{pot}}\right)$$

- Bewegungs-GL aus Euler-Lagrange Formalismus:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

(Aus Variation der Wirkung)

Welche Dimension hat die Lagranedichte $\mathcal{L}$ in natürlichen Einheiten?

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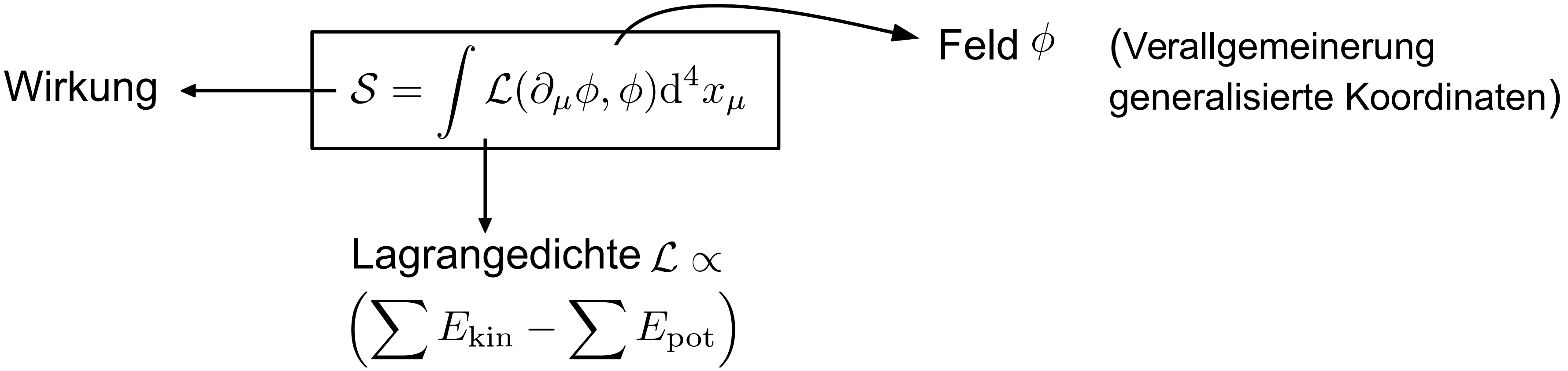

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(Aus Variation der Wirkung)

Welche Dimension hat die Lagranedichte $\mathcal{L}$ in natürlichen Einheiten?

$[\mathcal{L}] = \mathrm{GeV}^4$

Lagrangedichte für freie Bosonen und Fermionen

Für **Bosonen**:

$$\boxed{\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^*}$$

Für **Fermionen**:

$$\boxed{\mathcal{L} = \overline{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi}$$

- Beweis durch Anwendung der Euler-Lagrange Gleichung (→ hier für Bosonen):

$$\begin{tikzcd}
\partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^*)} \arrow[d] & - & \frac{\partial \mathcal{L}}{\partial \phi^*} \arrow[d] & = 0 & \\
\partial^\mu \partial_\mu \phi & & -m^2 \phi \arrow[rr] & & \left(\partial^\mu \partial_\mu + m^2\right)\phi = 0
\end{tikzcd}$$

- **Anmerkung:** bei der Variation sind die Felder ϕ^*, ϕ, $\overline{\psi}$, ψ als unabhängig voneinander zu betrachten.

Globale & Lokale Phasentransformationen

- Lagrangedichte kovariant unter **globalen** **Phasentransformationen** (→ hier für Fermionen):

$$\boxed{\begin{aligned}\psi(\vec{x},t) &\to \psi'(\vec{x},t) = e^{i\vartheta}\psi(\vec{x},t)\\ \overline{\psi}(\vec{x},t) &\to \overline{\psi'}(\vec{x},t) = \overline{\psi}(\vec{x},t)e^{-i\vartheta}\end{aligned}}$$

$$\vartheta \neq \vartheta(\vec{x},t)$$

$$\mathcal{L}' = \overline{\psi'}\left(i\gamma^\mu\partial_\mu - m\right)\psi' = \overline{\psi}e^{-i\vartheta}\left(i\gamma^\mu\partial_\mu - m\right)e^{i\vartheta}\psi$$

$$= \overline{\psi}\left(i\gamma^\mu\ \partial_\mu \qquad - m\right)\psi \quad = \mathcal{L}$$

- Phase ϑ and jedem Punkt $\vec{x}$ und zu jeder Zeit t fest vorgegeben.
- Was passiert wenn an jedem Punkt in $(\vec{x},t)$ eine andere Phase erlaubt ist?

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$$\vartheta = \vartheta(\vec{x},t)$$

Ableitung verbindet benachbarte Punkte in $(\vec{x},t)$

$$\partial_\mu \longrightarrow \frac{\psi(x+\Delta x)-\psi(x)}{\Delta x}$$

$$\mathcal{L}' = \overline{\psi'}\left(i\gamma^\mu\partial_\mu - m\right)\psi' = \overline{\psi}e^{-i\vartheta}\left(i\gamma^\mu\partial_\mu - m\right)e^{i\vartheta}\psi$$

$$= \overline{\psi}\left(i\gamma^\mu(\partial_\mu + i\partial_\mu\vartheta) - m\right)\psi \quad \neq \mathcal{L}$$

bricht Kovarianz

- Phase ϑ and jedem Punkt $\vec{x}$ und zu jeder Zeit t fest vorgegeben.
- Was passiert wenn an jedem Punkt in $(\vec{x},t)$ eine andere Phase erlaubt ist?

Globale & Lokale Phasentransformationen

- Lagrangedichte “kovariant” unter **globalen & lokalen Phasentransformationen** (→ hier für Fermionen):

$$\boxed{\begin{aligned} &\psi(\vec{x},t) \to \psi'(\vec{x},t) = e^{i\vartheta}\psi(\vec{x},t) \\ &\overline{\psi}(\vec{x},t) \to \overline{\psi'}(\vec{x},t) = \overline{\psi}(\vec{x},t)e^{-i\vartheta} \\ &D_\mu \to D'_\mu = D_\mu - i\partial_\mu\vartheta \end{aligned}}$$

$$\vartheta = \vartheta(\vec{x},t)$$

beliebiges Eichfeld

$$D_\mu \to D'_\mu = D_\mu - i\partial_\mu\vartheta \longrightarrow \partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$$

(Kovariante Ableitung)

$$\mathcal{L}' = \overline{\psi'}\left(i\gamma^\mu D'_\mu - m\right)\psi' = \overline{\psi}e^{-i\vartheta}\left(i\gamma^\mu(D_\mu - i\partial_\mu\vartheta) - m\right)e^{i\vartheta}\psi$$

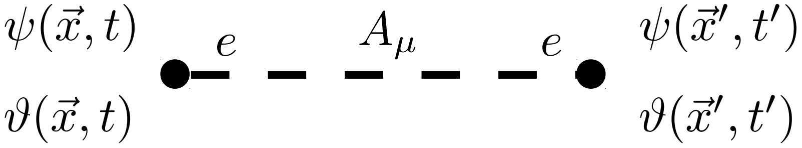
$$= \overline{\psi}\left(i\gamma^\mu(D_\mu - i\partial_\mu\vartheta + i\partial_\mu\vartheta) - m\right)\psi \quad = \mathcal{L}$$

- Transformationsverhalten des Eichfeldes

$A_\mu \to A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\vartheta$ ⟶ bekannt aus Elektrodynamik!

Zusammenfassung: Eichfelder

- Es ist möglich für das Feld $\psi(\vec{x},t)$ eine beliebige Phase $\vartheta(\vec{x},t)$ zu erlauben
- Erfordert Einführung eines vermittelnden Feldes A_μ, das diese Information von $(\vec{x},t)$ nach $(\vec{x}',t')$ transportiert

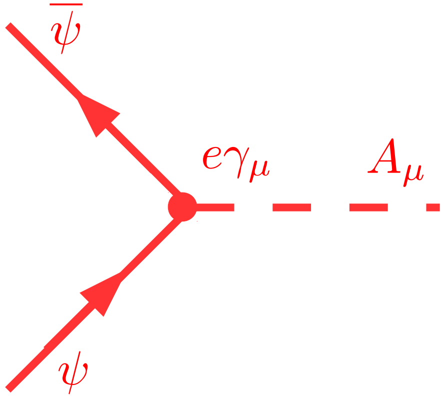


- Eichfeld A_μ koppelt an Größe e des Feldes $\psi(\vec{x},t)$, die mit elektrischer Ladung identifiziert werden kann

Das wechselwirkende Fermion

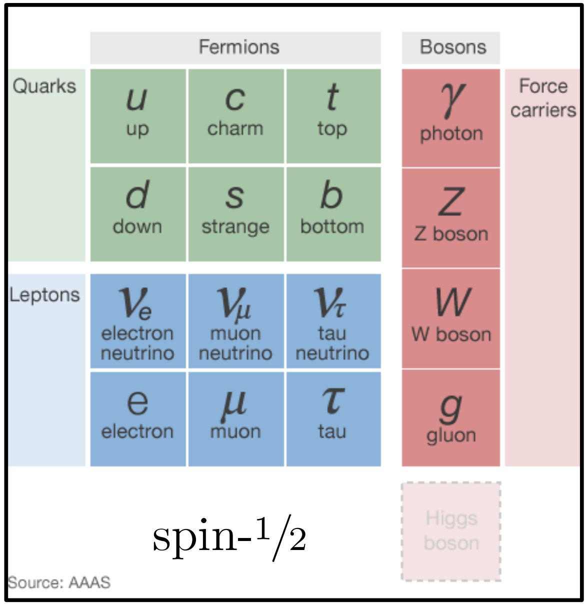
- Einführung der kovarianten Ableitung führt zu **Lagrangedichte für wechselwirkendes Fermion** mit Ladung e:

$$\boxed{\begin{aligned}\mathcal{L}_{\text{IA}} &= \overline{\psi}\,(i\gamma^\mu(D_\mu - m)\,\psi \\ &= \underbrace{\overline{\psi}\,(i\gamma^\mu \;\; \partial_\mu - m)\,\psi}_{\text{freies Fermionfeld}} - \underbrace{e\overline{\psi}\gamma^\mu A_\mu\psi}_{\text{WW-Term}}\end{aligned}}$$



- **Anmerkung:** hier nicht diskutiert – dynamischer Term zur Beschreibung eines “frei” propagierenden Eich(=Photon)feldes

Das Standardmodell der Teilchenphysik

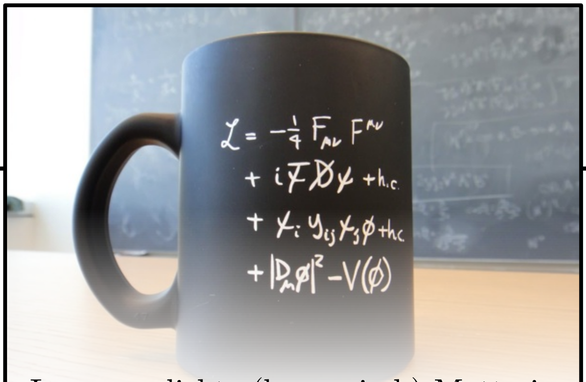


	Fermions			Bosons	
Quarks	u up	c charm	t top	γ photon	Force carriers
	d down	s strange	b bottom	Z Z boson	
Leptons	ν_e electron neutrino	ν_μ muon neutrino	ν_τ tau neutrino	W W boson	
	e electron	μ muon	τ tau	g gluon	
				Higgs boson	

spin-$^1/_2$

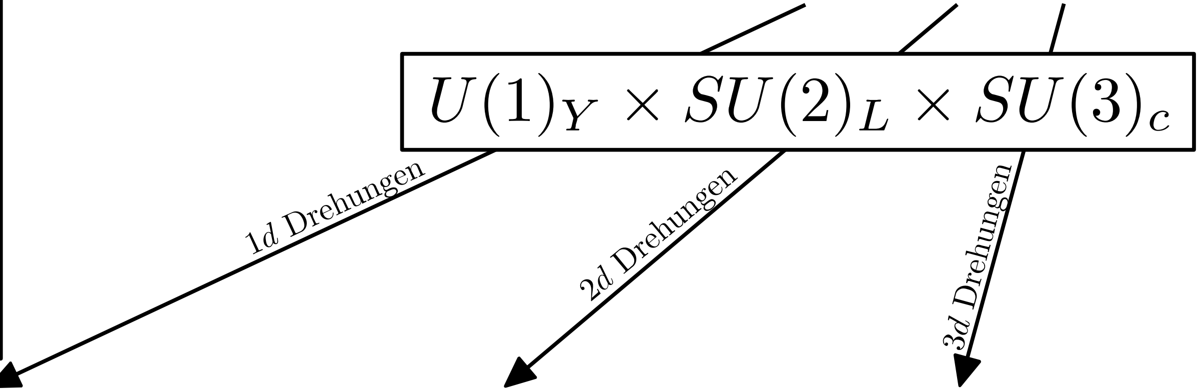
Source: AAAS

Beschreibung der ursprünglichen Struktur der uns umgebenden Natur.

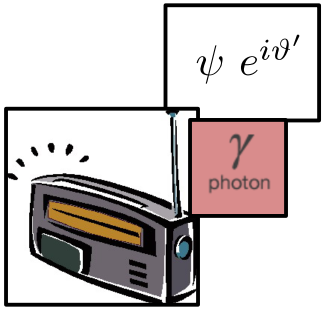


$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi} \not{D} \psi + h.c. + \psi_i y_{ij} \psi_j \phi + h.c. + |D_\mu \phi|^2 - V(\phi)$$

Lagrangedichte (baryonisch) Matterie

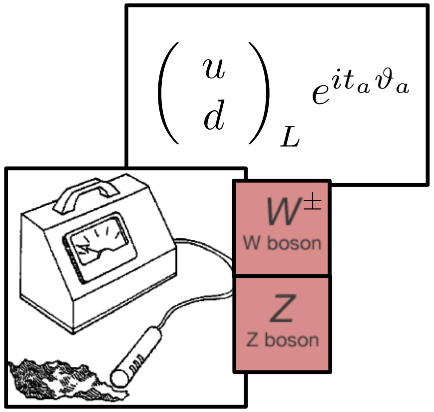


$$U(1)_Y \times SU(2)_L \times SU(3)_c$$



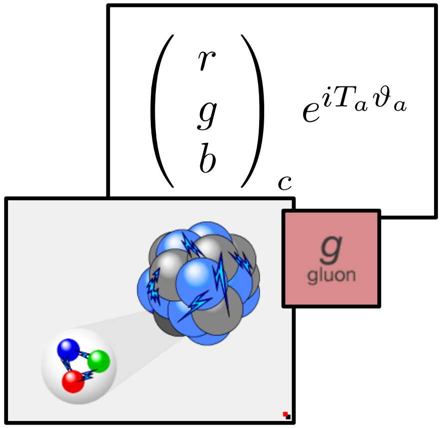
$$\psi\ e^{i\vartheta'}$$

Elektromagnetismus



$$\begin{pmatrix} u \\ d \end{pmatrix}_L e^{it_a\vartheta_a}$$

Schwache Kraft



$$\begin{pmatrix} r \\ g \\ b \end{pmatrix}_c e^{iT_a\vartheta_a}$$

Starke Kraft

- Bezieht Erklärungs-/Vorhersagekraft aus Anwendung von Symmetrien!
- Kräfte ↔ masselose Vermittlerteilchen.

A wealth of structures

$$\mathcal{L}^{\rm SM} = \mathcal{L}^{\rm Lepton}_{\rm kin} + \mathcal{L}^{CC}_{\rm IA} + \mathcal{L}^{NC}_{\rm IA} + \mathcal{L}^{\rm Gauge}_{\rm kin} + \mathcal{L}^{\rm Higgs}_{kin} + \mathcal{L}^{\rm Higgs}_{V(\phi)} + \mathcal{L}^{\rm Higgs}_{\rm Yukawa}$$

$$\mathcal{L}^{\rm Lepton}_{\rm kin} = i\bar{e}\gamma^\mu\partial_\mu e + i\bar{\nu}\gamma^\mu\partial_\mu\nu$$

$$\mathcal{L}^{CC}_{\rm IA} = -\frac{e}{\sqrt{2}\sin\theta_W}\left[W^+_\mu\bar{\nu}\gamma_\mu e_L + W^-_\mu\bar{e}_L\gamma_\mu\nu\right]$$

$$\mathcal{L}^{NC}_{\rm IA} = -\frac{e}{2\sin\theta_W\cos\theta_W}Z_\mu\left[(\bar{\nu}\gamma_\mu\nu) + (\bar{e}_L\gamma_\mu e_L)\right]\boxed{- e\left[A_\mu + \tan\theta_W Z_\mu\right](\bar{e}\gamma_\mu e)}$$

$$\mathcal{L}^{\rm Gauge}_{\rm kin} = -\frac{1}{2}Tr\left(W^a_{\mu\nu}W^{a\mu\nu}\right) - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}\Bigg|$$

$$B_\mu \rightarrow A_\mu$$
$$W^3_\mu \rightarrow Z_\mu$$

$$\mathcal{L}^{\rm Higgs}_{\rm kin} = \frac{1}{2}\partial_\mu H\partial^\mu H + \left(1 + \frac{1}{v}\frac{H}{\sqrt{2}}\right)^2 m_W^2 W^+_\mu W^{\mu-} + \left(1 + \frac{1}{v}\frac{H}{\sqrt{2}}\right)^2 m_Z^2 Z_\mu Z^\mu$$

$$\mathcal{L}^{\rm Higgs}_{V(\phi)} = -\frac{m_H^2 v^2}{4} + \frac{m_H^2}{2}\left(\frac{H}{\sqrt{2}}\right)^2 + \frac{m_H^2}{v}\left(\frac{H}{\sqrt{2}}\right)^3 + \frac{m_H^2}{4v^2}\left(\frac{H}{\sqrt{2}}\right)^4$$

$$\mathcal{L}^{\rm Higgs}_{\rm Yukawa} = -\left(1 + \frac{1}{v}\frac{H}{\sqrt{2}}\right)m_e\bar{e}e$$

Full SM Lagrangian density (first lepton generation)

- “Simple” (local) symmetry requirements on $\mathcal{L}$ enforce complex interactions.

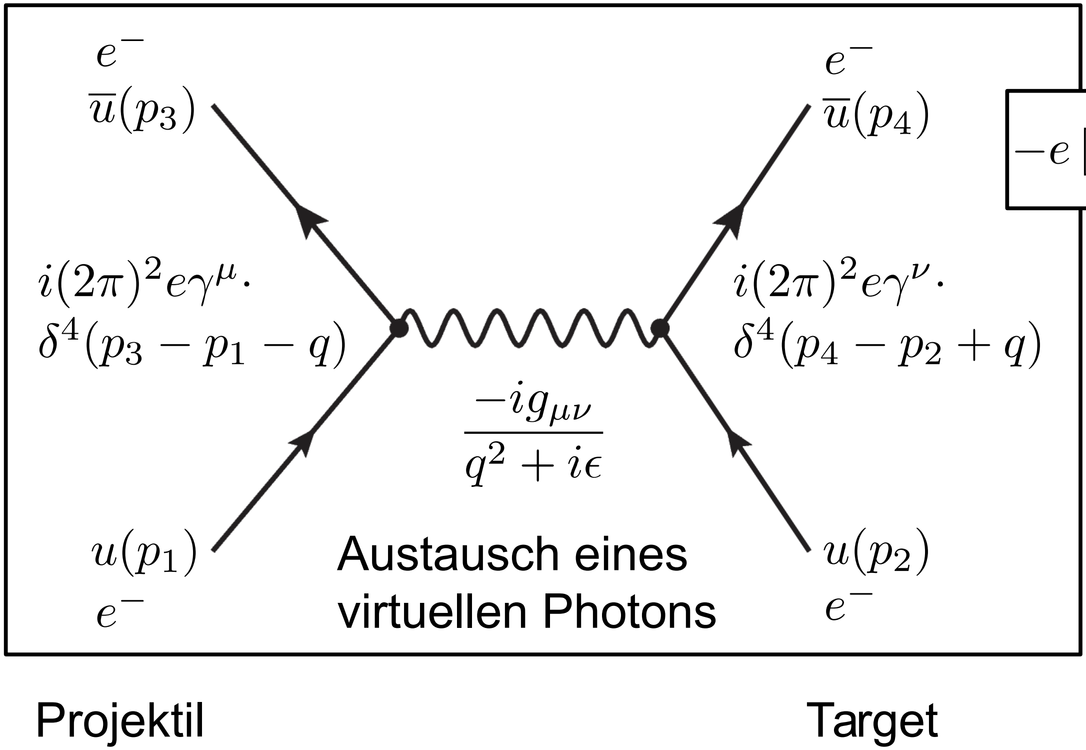
Feynman Regeln

Richard Feynman

(*11. Mai 1918, † 15. Februar 1988)

Feynman Regeln (der QED)

- Wechselwirkungsterme in der Lagrangedichte lassen sich in **bildliche Regeln** zur Berechnung von Wirkungsquerschnitten übersetzen.



$$\boxed{-e\left[\underset{\searrow}{A_\mu} + \tan\theta_W \underset{\searrow}{Z_\mu}\right](\overline{e}\gamma_\mu e)}$$

Austausch Photon

Austausch Z-Boson

Vollständige Ableitung siehe Backup

$$\mathcal{S}_{fi}^{(1)} = i\left((2\pi)^2 e\right)^2 \cdot \int \mathrm{d}^4 q\ \delta^4(p_3 - p_1 - q)\overline{u}(p_3)\gamma^\mu u(p_1)\frac{-g_{\mu\nu}}{q^2 + i\epsilon}\delta^4(p_4 - p_2 + q)\overline{u}(p_4)\gamma^\nu u(p_2)$$

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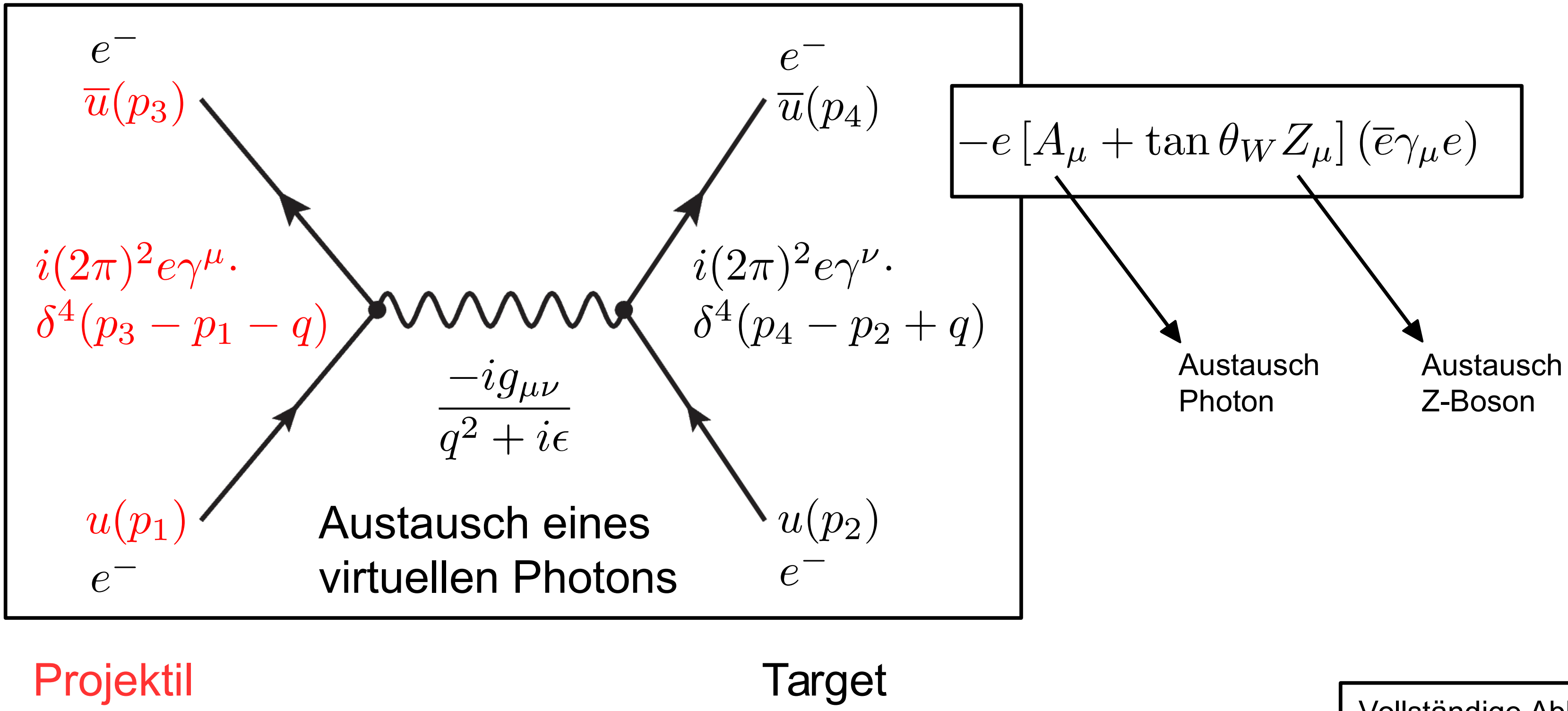

$$-e\left[A_\mu + \tan\theta_W Z_\mu\right](\overline{e}\gamma_\mu e)$$

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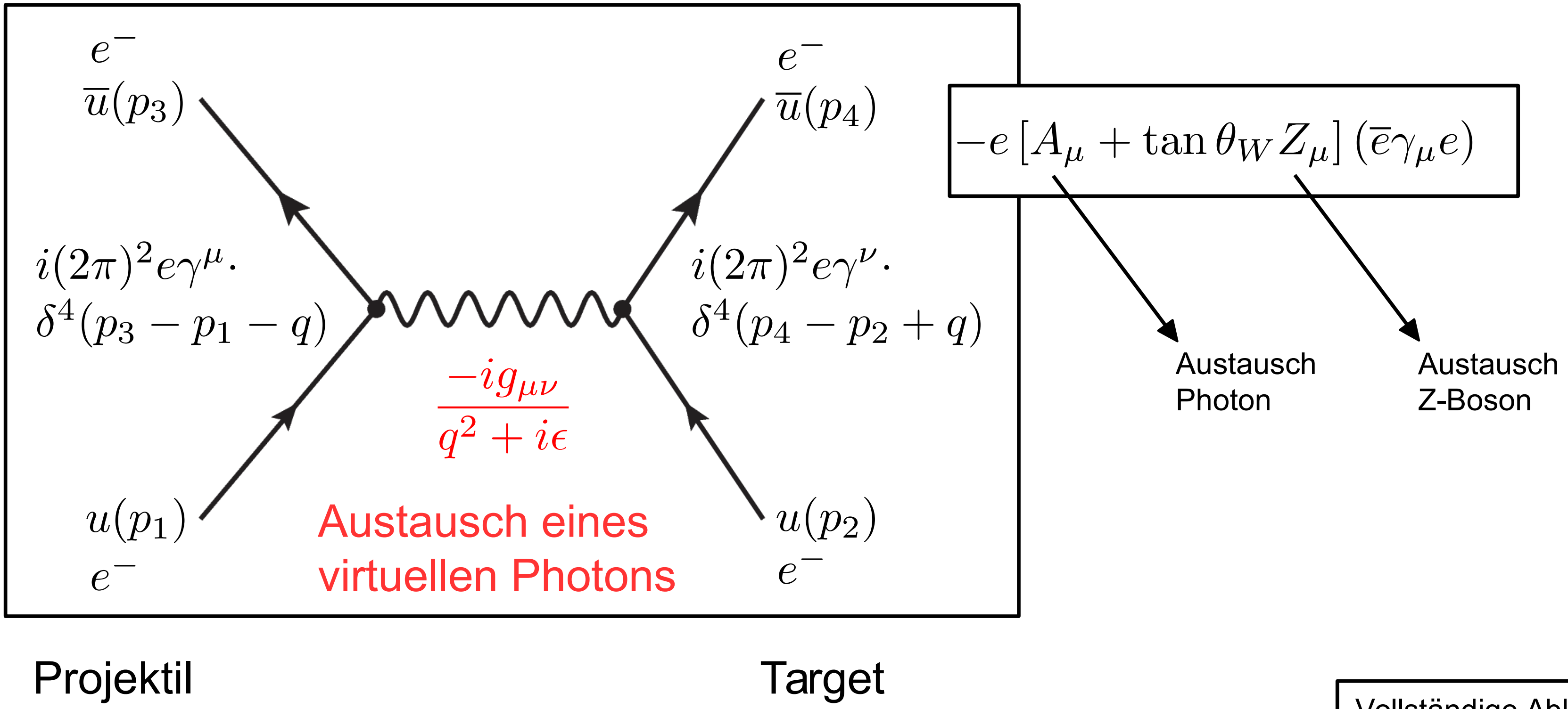

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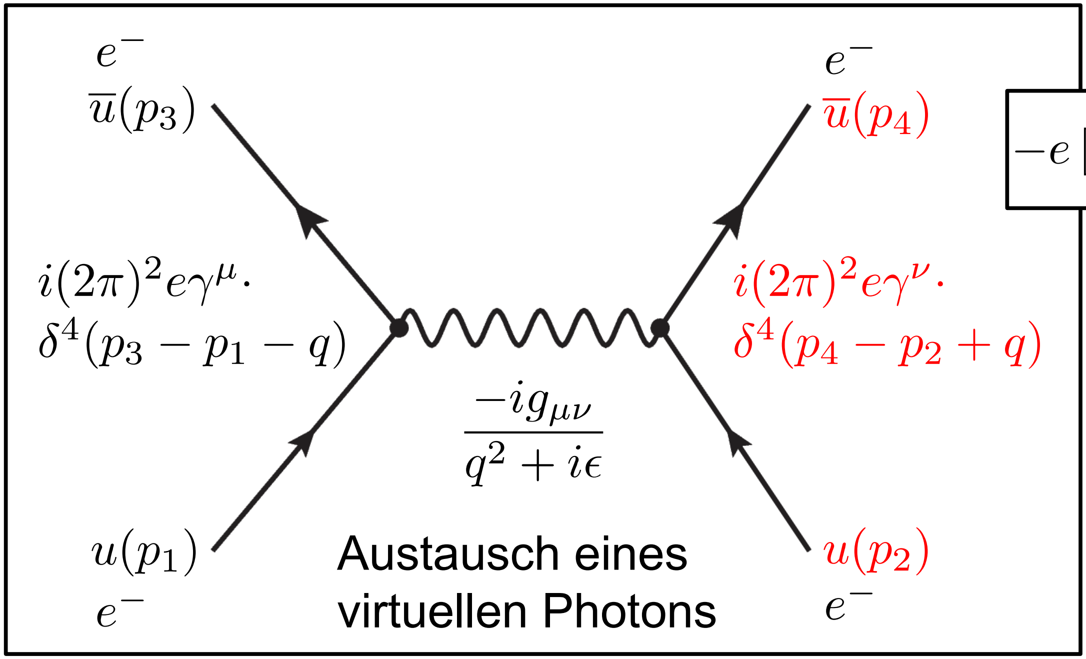

Projektil

Target

$$\boxed{-e\left[A_\mu + \tan\theta_W Z_\mu\right](\overline{e}\gamma_\mu e)}$$

Austausch Photon

Austausch Z-Boson

Vollständige Ableitung siehe Backup

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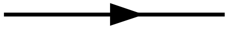
Feynman Regeln (der QED)

- Feynman diagrams are a way to represent the elements of the matrix element calculation:

Legs:		
	$u(p)$ $(\overline{u}(p))$	• Incoming (outgoing) fermion.
	$\epsilon_\mu(k)$ $(\epsilon^*_\mu(k))$	• Incoming (outgoing) photon.
Vertices:		
	$i(2\pi)^2 e\gamma^\mu \cdot \delta^4(p_f - p_i - q)$	• Lepton-photon vertex.
Propagators:		
	$\frac{i(\gamma^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon}$	• Fermion propagator.
	$\frac{-ig^{\mu\nu}}{q^2 + i\epsilon}$	• Photon propagator.
Four-momenta of all virtual particles have to be integrated out.		

Gliederung der Vorlesung

YOU ARE HERE

Backup

Properties of $\vec{\alpha}$ and β

- Operators $\vec{\alpha}$ and β can be expressed by matrices:

  Must be hermitian: $\hat{H}_0$ should have real *eigenvalues*.

Properties of $\vec{\alpha}$ and β

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Must be hermitian: $\hat{H}_0$ should have real *eigenvalues*.

Must be traceless:

$$\boxed{Tr(\alpha_i) = Tr(\alpha_i \overset{\overset{\mathbb{I}}{\uparrow}}{\beta\beta}) \underset{\substack{\downarrow\\ \textit{cyclic}\\ \textit{permutation}}}{=} Tr(\beta\alpha_i\beta) \underset{\substack{\downarrow\\ \textit{anti-commutator}\\ \textit{relation}}}{=} -Tr(\beta\beta\alpha_i) = -Tr(\alpha_i) = 0}$$

Properties of $\vec{\alpha}$ and β

- Operators $\vec{\alpha}$ and β can be expressed by matrices:

Must have at least dim=4:

- $\alpha_i^2 = \mathbb{I}$ → has only eigenvectors ±1.
- $\beta^2 = \mathbb{I}$ → has only eigenvectors ±1.
- Dimension must be even to obtain 0 trace.
- $\mathbb{I}$ + Pauli matrices $(\mathbb{I}, \sigma_i)$ form a basis of the space of 2×2 matrices. But $\mathbb{I}$ is not traceless (→ no chance to obtain four independent(!) traceless matrices).
- Simplest representation must at least have dim=4 (can be higher dimensional though).

The transformation behavior of spinors

$$\boxed{\begin{array}{lll} \Lambda : x^{\mu} \rightarrow x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} & & \text{Lorentz vector} \\ \quad \psi_{\alpha}(x) \rightarrow \psi'_{\alpha}(x') = S_{\alpha\beta}(\Lambda)\psi_{\beta}(\Lambda x) & & \text{Spinor} \end{array}}$$

(*Lorentz transformation*)

mixes components of ψ

acts on coordinates

- How does $S(\Lambda)$ look like?

$$S(\Lambda) = e^{-\frac{i}{4} r_{\mu\nu}\sigma^{\mu\nu}} = \begin{cases} e^{-i\vec{\varphi}\cdot\left(\frac{1}{2}\vec{\Sigma}\right)} \xrightarrow{\text{Spacial rotation}} \cos\left(\frac{\varphi}{2}\right) - i\sin\left(\frac{\varphi}{2}\right)\left(\hat{\varphi}\cdot\vec{\Sigma}\right) \\ e^{\hat{v}\cdot\frac{1}{2}\vec{\alpha}} \xrightarrow{\text{Boost}} \cosh\left(\frac{v}{2}\right) + \sinh\left(\frac{v}{2}\right)\left(\hat{v}\cdot\vec{\alpha}\right) \end{cases}$$

Rotation of 2π around spacial quantization axis turns $\psi_{\alpha}(x) \rightarrow -\psi_{\alpha}(x)$.

QM model of particle scattering

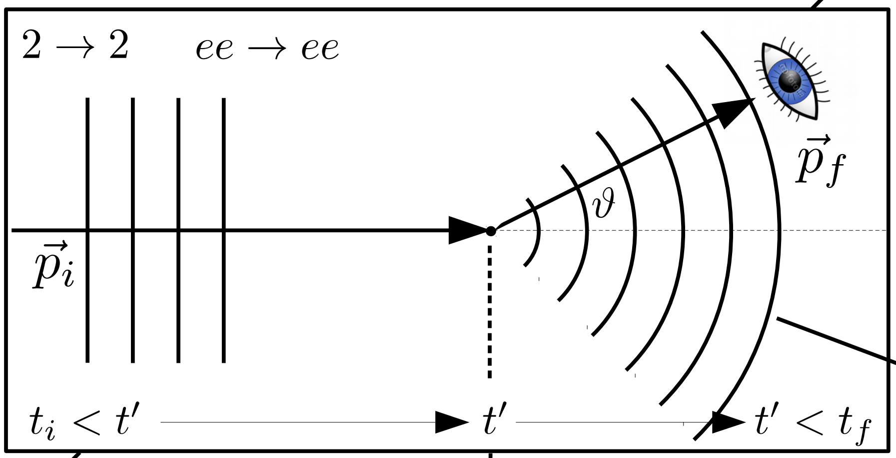
- Consider incoming collimated beam of projectile particles on a target particle:

Scattering matrix $\mathcal{S}$ transforms initial state wave function ϕ_i into scattering wave ψ_{scat} ($\psi_{\text{scat}} = \mathcal{S} \cdot \phi_i$).

Observation (in $\Delta\Omega$): projection of plain wave ϕ_f out of spherical scattering wave ψ_{scat}.

Observation probability:

$$\mathcal{S}_{fi} = \phi_f^\dagger \cdot \psi_{\text{scat}}$$
$$= \phi_f^\dagger \cdot \mathcal{S} \cdot \phi_i$$



Spherical scattering wave ψ_{scat}.

Initial particle: described by plain wave ϕ_i.

Localized potential.

Solution for ψ_{scat}

- In the case of fermion scattering the scattering wave ψ_{scat} is obtained as a solution of the inhomogeneous *Dirac* equation for an interacting field:

$$\boxed{\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi_{\mathrm{scat}} = -e\gamma^{\mu}A_{\mu}\psi_{\mathrm{scat}}} \quad (+)$$

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$$(i\gamma^{\mu}\partial_{\mu} - m)\,K(x-x') = \delta^{4}(x-x')$$

$$\psi_{\mathrm{scat}}(x) = -e\int K(x-x')\gamma^{\mu}A_{\mu}(x')\psi_{\mathrm{scat}}(x')\mathrm{d}^{4}x'$$

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$$(i\gamma^\mu\partial_\mu - m)\,\psi_{\text{scat}}(x) = -e\int \underbrace{(i\gamma^\mu\partial_\mu - m)\,K(x-x')}_{\delta^4(x-x')}\gamma^\mu A_\mu(x')\psi_{\text{scat}}(x')\mathrm{d}^4x'$$

$$= -e\gamma^\mu A_\mu(x)\psi_{\text{scat}}(x)$$

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- This is not a solution to (+), since ψ_{scat} appears on the left- and on the right-hand side of the equation. It turns the differential equation into an integral equation. It propagates the solution from the point x' to x.

Green's function in *Fourier* space

- The best way to find the *Green's* function is to go to the *Fourier* space:

$$K(x-x') = (2\pi)^{-4} \int \tilde{K}(p) e^{-ip(x-x')} \mathrm{d}^4 p \qquad \text{(\textit{Fourier} transform)}$$

Applying the *Dirac* equation to the *Fourier* transform of $K(x-x')$ turns the derivative into a product operator:

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From the uniqueness of the *Fourier* transformation the solution for $\tilde{K}(p)$ follows:

$$(\gamma^\mu p_\mu - m)\,\tilde{K}(p) = \mathbb{I}_4$$

Fermion propagator

- The *Fourier* transform of the *Green's* function is called *fermion* propagator:

$$(\gamma^\mu p_\mu - m)\,\tilde{K}(p) = \mathbb{I}_4$$

$$(\gamma^\mu p_\mu + m)\cdot(\gamma^\mu p_\mu - m)\,\tilde{K}(p) = (\gamma^\mu p_\mu + m)\cdot\mathbb{I}_4$$

$$\boxed{\tilde{K}(p) = \frac{(\gamma^\mu p_\mu + m)}{p^2 - m^2}}$$

(fermion propagator)

- The fermion propagator is a 4×4 matrix, which acts in the *Spinor* space.
- It is only defined for virtual fermions since $p^2 - m^2 = E^2 - \vec{p}^2 - m^2 \neq 0$.

Fermion propagator ↔ *Green*'s function

- The *Green's* function can be obtained from the propagator by inverse *Fourier* transformation:

$$K(x-x') = (2\pi)^{-4} \int \mathrm{d}^3\vec{p}\, e^{i\vec{p}(\vec{x}-\vec{x}')} \int_{-\infty}^{+\infty} \mathrm{d}p_0 \frac{(\gamma^\mu p_\mu + m)}{(p_0 - E)(p_0 + E)}\; e^{-ip_0(t-t')}$$

$$\downarrow$$

$$E = \sqrt{\vec{p}^2 + m^2}$$

- This integral can be solved with the methods of *function theory*.

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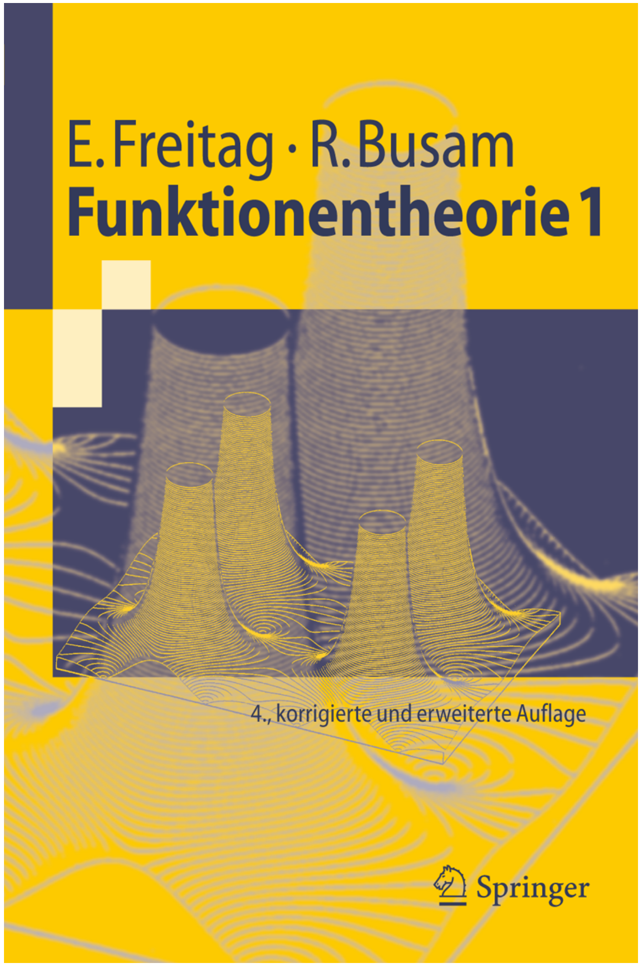
$$\downarrow$$

$$E = \sqrt{\vec{p}^2 + m^2}$$

- This integral can be solved with the methods of *function theory*.

- $K(x-x')$ has two poles in the integration plane (at $p_0 = \pm E$).

Excursion into function theory



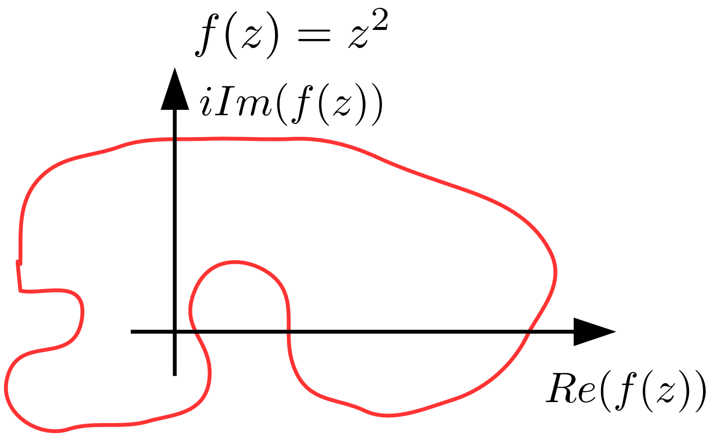
E. Freitag · R. Busam
Funktionentheorie 1
4., korrigierte und erweiterte Auflage
Springer

cf. Freitag/Busam Funktionentheorie

Residual theorem

- When integrating a “well behaved” function w/o poles in the complex plain the path integral along any closed path $\mathcal{C}$ is 0:

Example: $\oint_{\mathcal{C}} z^2 \mathrm{d}z = 0$



- When integrating a “well behaved” function w/ poles in the complex plain the solution is $2\pi i \times$ the sum of “residuals” of the poles surrounded by the path:

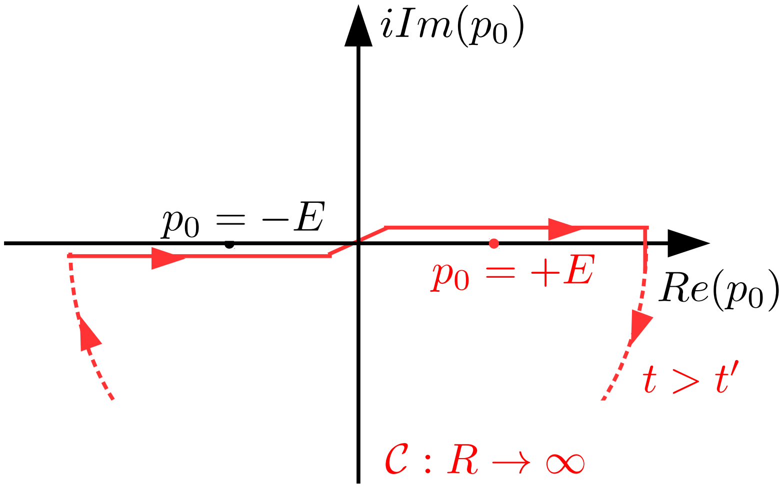
Example: $\oint_{\mathcal{C}} \frac{R}{z} \mathrm{d}z = 2\pi i \times R$

No matter how $\mathcal{C}$ is chosen, as long as it includes $z = (0 + i0)$.

The *Green*'s function (time integration for $t > t'$)

- Choose path $\mathcal{C}$ in complex plain to circumvent poles:

$$\boxed{\int_{-\infty}^{+\infty} \mathrm{d}p_0 \frac{(\gamma^\mu p_\mu + m)}{(p_0 - E)(p_0 + E)}\, e^{-ip_0(t-t')}}$$



- For $t > t'$ ($e^{-ip_0(t-t')} \to 0$ for $Im(p_0) \ll 0$):
  → close contour in lower plane & calculate integral from residual of enclosed pole.

$$\oint_{\mathcal{C}} \mathrm{d}p_0 \underbrace{\frac{1}{p_0 - E}}_{\substack{\text{pole at:} \\ p_0 = +E}} \cdot \underbrace{\frac{(\gamma^\mu p_\mu + m)}{p_0 + E}\, e^{-ip_0(t-t')}}_{\text{residual: } f(p_0)} = -2\pi i \cdot f(p_0)\big|_{p_0=+E}$$

Sign due to sense of integration.

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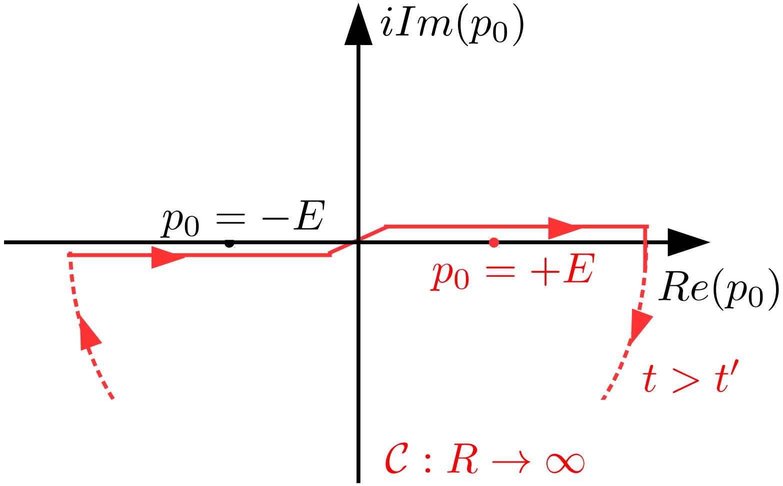

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$$\boxed{K(x - x') = -i(2\pi)^{-3} \int \mathrm{d}^3\vec{p}\, \frac{+\gamma^0 E - \vec{\gamma}\vec{p} + m}{2E} \cdot e^{-iE(t-t') + i\vec{p}(\vec{x}-\vec{x}')}}$$

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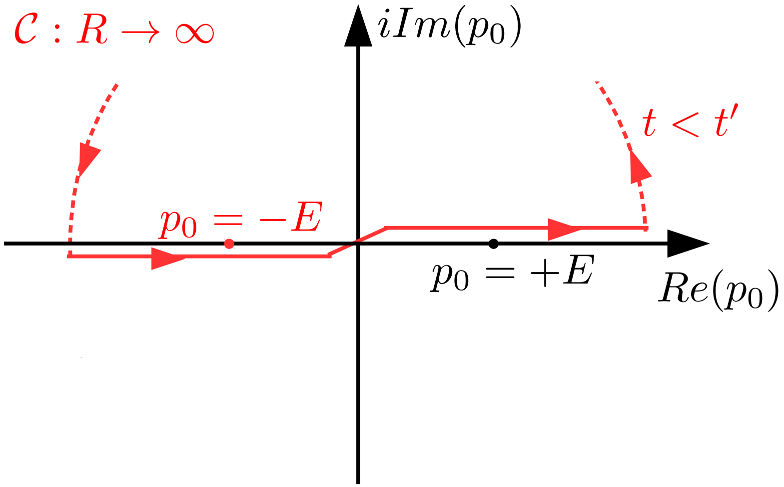

- For $t < t'$ ($e^{+ip_0(t-t')} \to 0$ for $Im(p_0) \gg 0$):
  → close contour in upper plane & calculate integral from residual of enclosed pole.

$$\oint_{\mathcal{C}} \mathrm{d}p_0 \underbrace{\frac{1}{p_0 + E}}_{\substack{\text{pole at:}\\ p_0 = -E}} \cdot \underbrace{\frac{(\gamma^\mu p_\mu + m)}{p_0 - E}\, e^{-ip_0(t-t')}}_{\text{residual: } f(p_0)} = +2\pi i \cdot f(p_0)\big|_{p_0=-E}$$

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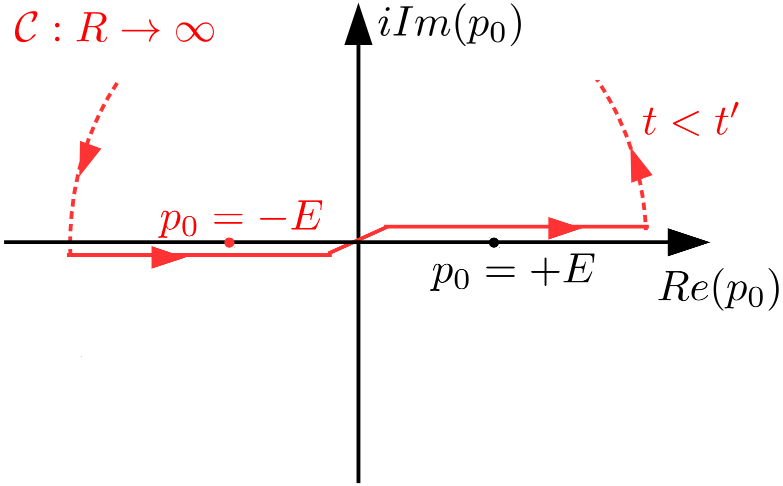

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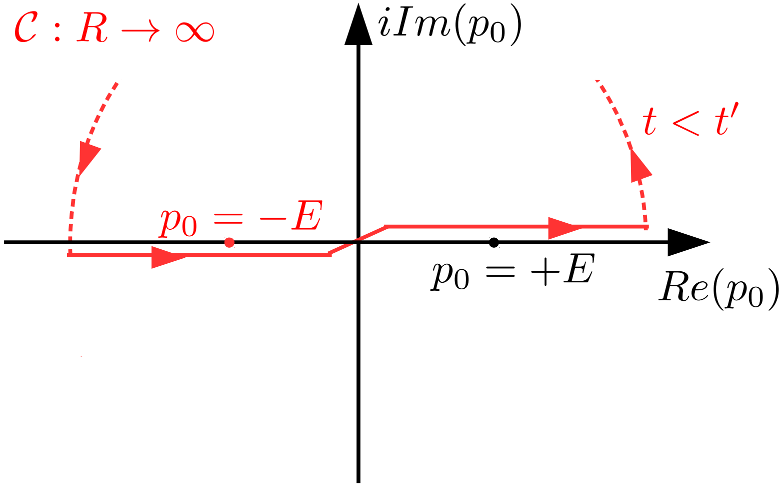

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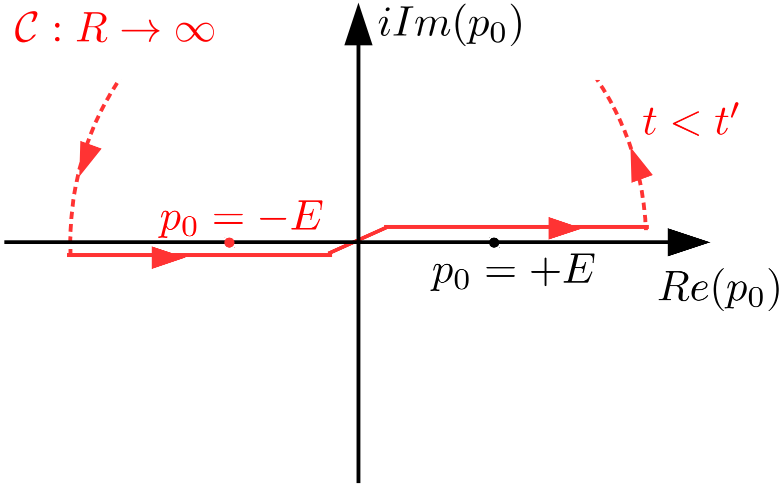
Sign due to $p_0 = -E$ in integral kernel.

The *Green*'s function (Nota Bene)

- Choose path $\mathcal{C}$ in complex plain to circumvent poles:

$$\boxed{\int_{-\infty}^{+\infty} \mathrm{d}p_0 \frac{(\gamma^\mu p_\mu + m)}{(p_0 - E)(p_0 + E)}\; e^{-ip_0(t-t')}}$$



- The bending of the integration path can be avoided by shifting the poles by ϵ.

$$\left[p_0 + \left(E - \frac{i\epsilon}{2E}\right)\right] \cdot \left[p_0 - \left(E - \frac{i\epsilon}{2E}\right)\right] = p_0^2 - (\vec{p}^2 + m^2) + i\epsilon$$
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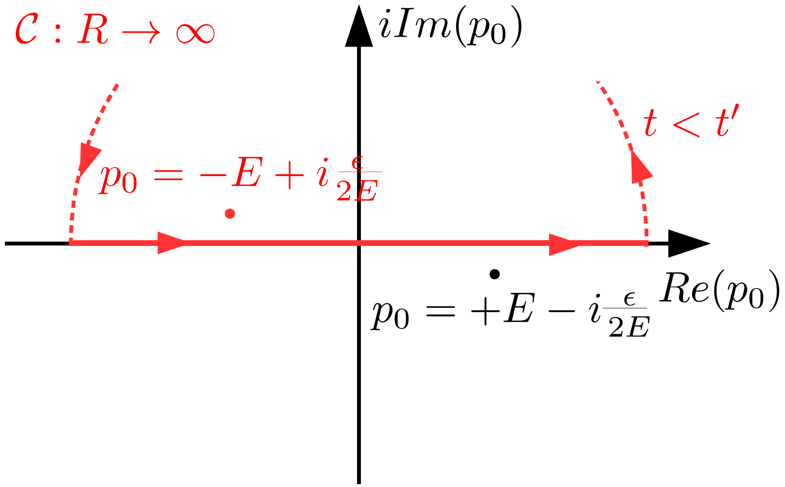

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$$\Big[\underset{\substack{\big\downarrow \\ (-E+i\frac{\epsilon}{2E})}}{p_0} + \Big(E - \frac{i\epsilon}{2E}\Big)\Big] \cdot \Big[\underset{\substack{\big\downarrow \\ (+E-i\frac{\epsilon}{2E})}}{p_0} - \Big(E - \frac{i\epsilon}{2E}\Big)\Big] = p_0^2 - (\vec{p}^2 + m^2) + i\epsilon$$

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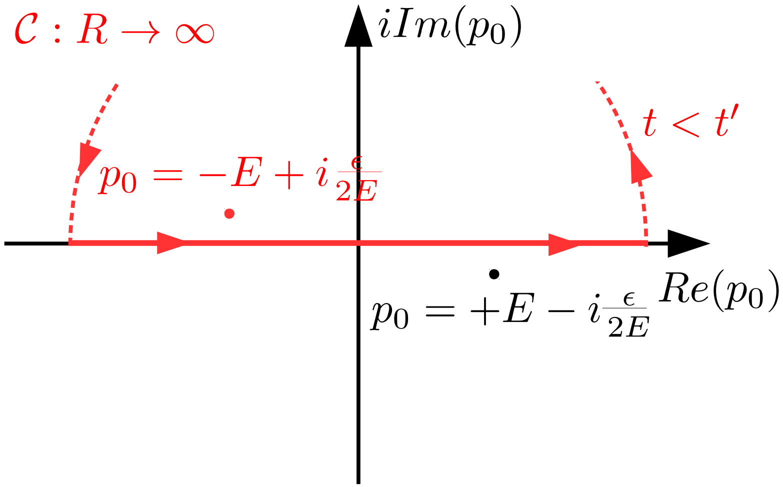

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$$\boxed{\tilde{K}(p) = \frac{(\gamma^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon} \qquad \epsilon > 0}$$

(fermion propagator)

Summary of time evolution

$$\tilde{K}(p) = \frac{(\gamma^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon} \qquad \epsilon > 0$$

(Fermion propagator in momentum space)

- *Green's* function (for $t > t'$, forward evolution):

$$K(x - x') = -i(2\pi)^{-3} \int \mathrm{d}^3\vec{p}\, \frac{+\gamma^0 E - \vec{\gamma}\vec{p} + m}{2E} \cdot e^{-iE(t-t') + i\vec{p}(\vec{x}-\vec{x}')}$$

- *Green's* function (for $t < t'$, backward evolution):

$$K(x - x') = -i(2\pi)^{-3} \int \mathrm{d}^3\vec{p}\, \frac{-\gamma^0 E - \vec{\gamma}\vec{p} + m}{2E} \cdot e^{+iE(t-t') + i\vec{p}(\vec{x}-\vec{x}')}$$

- But why did I choose explicitly THIS integration path and not another one?

Summary of time evolution

$$\tilde{K}(p) = \frac{(\gamma^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon} \qquad \epsilon > 0$$

(Fermion propagator in momentum space)

- The chosen integration path defines the time evolution of the solution.
- General solution to (inhomogeneous) Dirac equation:

$$\phi(t,\vec{x}) = \begin{cases} i\int \mathrm{d}^3\vec{x}'K(x-x')\gamma^0\phi(t',\vec{x}') & \text{for } t > t' \\ 0 & \text{for } t < t' \end{cases}$$

particle w/ pos. energy traveling forward in time.

$$\overline{\phi}(t,\vec{x}) = \begin{cases} 0 & \text{for } t > t' \\ i\int \mathrm{d}^3\vec{x}'\overline{\phi}(t',\vec{x}')\gamma^0 K(x-x') & \text{for } t < t' \end{cases}$$

particle w/ pos. energy traveling backward in time.

$$\phi(t,\vec{x}) = \begin{cases} 0 & \text{for } t > t' \\ i\int \mathrm{d}^3\vec{x}'K(x-x')\gamma^0\phi(t',\vec{x}') & \text{for } t < t' \end{cases}$$

particle w/ neg. energy traveling forward in time.

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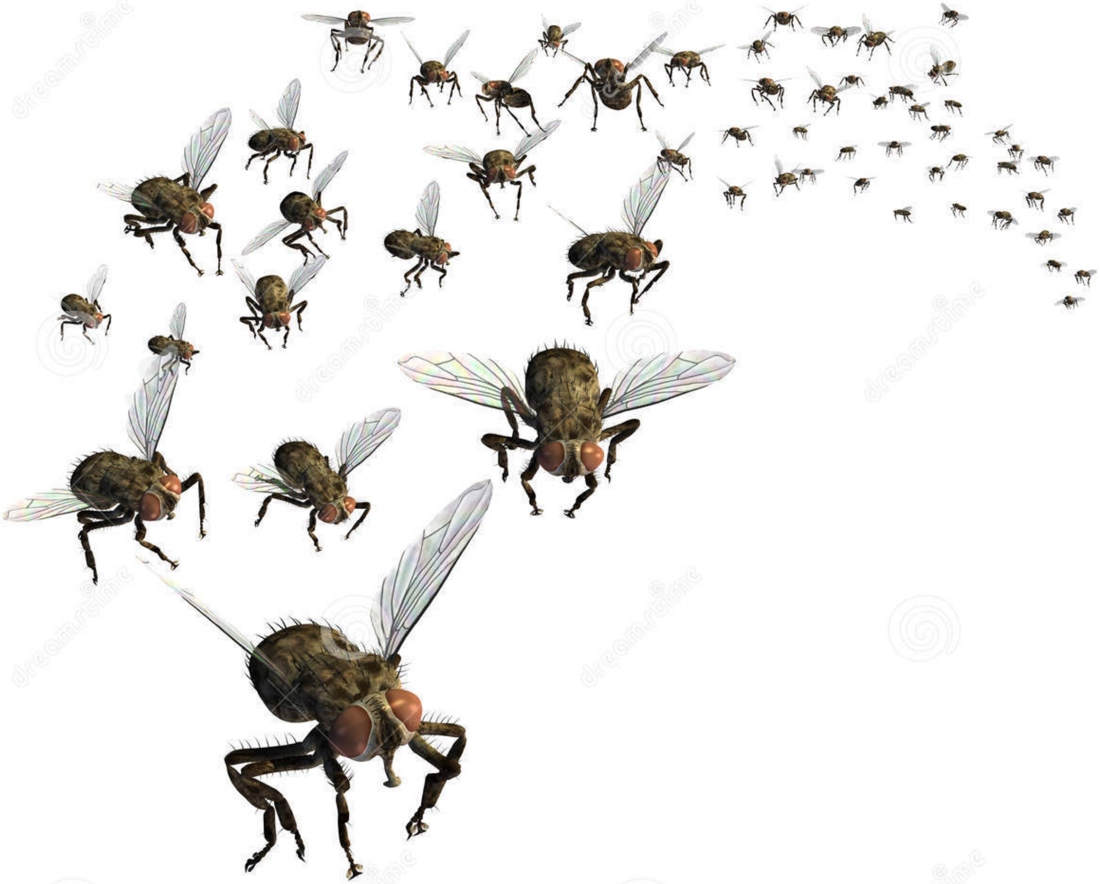
The perturbative series



The perturbative series

- The integral equation can be solved iteratively:

$$\boxed{\psi_{\text{scat}}(x) = \phi(x) - e\int K(x-x')\gamma^{\mu}A_{\mu}(x')\psi_{\text{scat}}(x')\mathrm{d}^4x'}$$

($\phi(x)$ = solution of the homogeneous *Dirac* equation)

- 0th order perturbation theory:

$$\psi^{(0)}(x) = \phi(x)$$

- Just take $\phi(x)$ as solution (→ boring).

- 1st order perturbation theory:

$$\psi^{(1)}(x) = \psi^{(0)}(x) \\ -e\int K(x-x')\gamma^{\mu}A_{\mu}(x')\psi^{(0)}(x')\mathrm{d}^4x'$$

- Assume that $\psi^{(0)}(x)$ is close enough to actual solution on RHS.

- 2nd order perturbation theory:

$$\psi^{(2)}(x) = \psi^{(0)}(x) \\ -e\int K(x-x')\gamma^{\mu}A_{\mu}(x')\psi^{(1)}(x')\mathrm{d}^4x'$$

- Take $\psi^{(1)}(x)$ as better approximation at RHS to solve inhomogeneous equation.

The perturbative series

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- Assume that $\psi^{(0)}(x)$ is close enough to actual solution on RHS.

- 2nd order perturbation theory:

$$\psi^{(2)}(x) = \psi^{(0)}(x) - e\int K(x-x')\gamma^\mu A_\mu(x')\psi^{(0)}(x')\mathrm{d}^4x' + e^2\iint K(x-x')\gamma^\mu A_\mu(x')K(x'-x'')\gamma^\mu A_\mu(x'')\psi^{(0)}(x'')\mathrm{d}^4x'\mathrm{d}^4x''$$

The matrix element $\mathcal{S}_{fi}$

- $\mathcal{S}_{fi}$ is obtained from the projection of the scattering wave ψ_{scat} on $\phi_f = \phi(x_f)$:

$$\boxed{\begin{aligned}\mathcal{S}_{fi} &= \int \mathrm{d}^4 x_f \phi_f^\dagger(x_f)\psi_{\text{scat}}(x_f) = \int \mathrm{d}^4 x_f \phi_f^\dagger(x_f)\mathcal{S}\phi_i(x_f)\\ &= \delta_{fi} + \mathcal{S}_{fi}^{(1)} + \mathcal{S}_{fi}^{(2)} + \ldots\end{aligned}}$$

“LO” **“NLO”**

- 1st order perturbation theory:

$$\mathcal{S}_{fi}^{(1)} = -e\int \mathrm{d}^4 x' \underbrace{\int \mathrm{d}^4 x_f \phi_f^\dagger(x_f) \overbrace{K(x_f - x')\gamma^\mu A_\mu(x')\phi_i(x')}^{\equiv \phi_f(x_f) = \phi(x_f)}}_{\equiv -i\overline{\phi}_f(x') = -i\overline{\phi}(x_f)}$$

For $E > 0$ and $t_f > t'$ respectively.

$$\phi(x_f) = \quad -e\int \mathrm{d}^4 x' K(x_f - x')\gamma^\mu A_\mu(x')\phi(x')$$ cf. backup slide 52

$$\phi(x') = i\int \mathrm{d}^3\vec{x}_f\,\phi(x_f)\gamma^0 K(x' - x_f) = -i\int \mathrm{d}^3\vec{x}_f \phi(x_f)\gamma^0 K(x_f - x')$$ cf. backup slide 73

NB: the time integration has already been carried out for the backward evolution from t_f to t' to arrive at the equation of backup slide 73.

The matrix element $\mathcal{S}_{fi}$

- $\mathcal{S}_{fi}$ is obtained from the projection of the scattering wave ψ_{scat} on $\phi_f = \phi(x_f)$:

$$\boxed{\begin{aligned}\mathcal{S}_{fi} = \int \mathrm{d}^4x_f\phi_f^\dagger(x_f)\psi_{\text{scat}}(x_f) &= \int \mathrm{d}^4x_f\phi_f^\dagger(x_f)\mathcal{S}\phi_i(x_f)\\ &= \delta_{fi} + \underbrace{\mathcal{S}_{fi}^{(1)}}_{\text{“LO”}} + \underbrace{\mathcal{S}_{fi}^{(2)}}_{\text{“NLO”}} + \ldots\end{aligned}}$$

- 1st order perturbation theory:

$$\mathcal{S}_{fi}^{(1)} = -e\int \mathrm{d}^4x'\int \mathrm{d}^4x_f\phi_f^\dagger(x_f)K(x_f - x')\gamma^\mu A_\mu(x')\phi_i(x')$$

$$\boxed{\mathcal{S}_{fi}^{(1)} = i\cdot\int \mathrm{d}^4x'\underbrace{e\,\overline{\phi}_f(x')\gamma^\mu A_\mu(x')\phi_i(x')}_{} \qquad \text{(1st order matrix element)}}$$

This corresponds exactly to the IA term in $\mathcal{L}$ (to be compared with slide 25 in body of this lecture)

The photon propagator

- The evolution of A_μ happens according to the inhomogeneous wave equation of the photon field (in Lorentz gauge $\partial_\mu A^\mu = 0$)

$$\boxed{\Box A^\mu = eJ^\mu} \qquad \text{(++)}$$

- We solve (++) again formally via the *Green's* function $D^{\mu\nu}(x-x')$ with the property:

$$\boxed{\begin{aligned} &\Box D^{\mu\nu}(x-x') = g^{\mu\nu}\delta^4(x-x') \\ &A^\mu(x) = e\int \mathrm{d}^4x' D^{\mu\nu}(x-x') J_\nu(x') \end{aligned}}$$

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$$\boxed{\Box A^\mu(x) = e\int \mathrm{d}^4x' \underbrace{\Box D^{\mu\nu}(x-x')}_{g^{\mu\nu}\delta^4(x-x')} J_\nu(x') = eJ^\mu(x)}$$

Green's function in *Fourier* space (fast forward)

- Check for the concrete form of the *Green*'s function again first in *Fourier* space:

$D^{\mu\nu}(x-x') = (2\pi)^{-4}\int \mathrm{d}^4q\tilde{D}^{\mu\nu}(q)e^{-iq(x-x')}$ (*Fourier* transform)

In analogy to the fermion case the defining property of $D^{\mu\nu}(x-x')$ in *Fourier* space

$$\Box D^{\mu\nu}(x-x') = (2\pi)^{-4}\int \mathrm{d}^4q\ (-q^2)\tilde{D}^{\mu\nu}(q)e^{-iq(x-x')} \overset{!}{=}$$

$$= (2\pi)^{-4}\int \mathrm{d}^4q\ g^{\mu\nu}e^{-iq(x-x')} = g^{\mu\nu}\delta^4(x-x')$$

(omitting the discussion of integral paths) leads to

$$\boxed{\tilde{D}^{\mu\nu}(q) = \frac{-g^{\mu\nu}}{q^2+i\epsilon} \qquad \epsilon > 0}$$

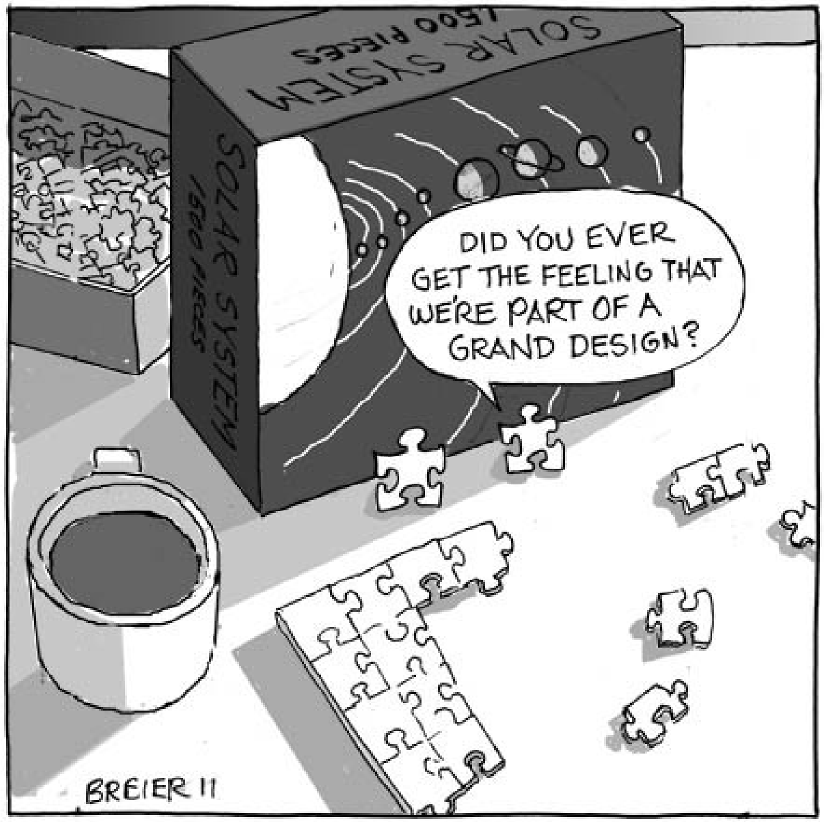
(photon propagator)

Green's function in *Fourier* space (fast forward)

- The *Green*'s function can again be obtained from the inverse *Fourier* transform.

$$D^{\mu\nu}(x-x') = (2\pi)^{-4} \int \mathrm{d}^4 q \frac{-g^{\mu\nu}}{q^2+i\epsilon} e^{-iq(x-x')}$$

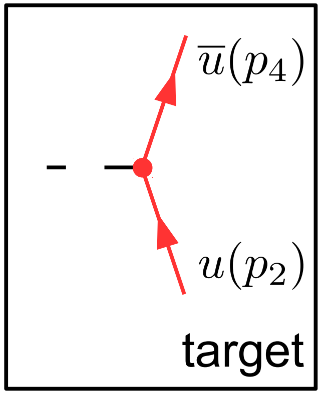
- We have now collected all pieces of the puzzle to complete the cross section calculation.



On the way to completion...

- *Ansatz* for target current:

$$\overline{\psi}_f(x'') = \overline{u}(p_4)e^{ip_4x''} \quad \psi_i(x'') = u(p_2)e^{-ip_2x''}$$
$$eJ^\nu(x'') = e\cdot\overline{\psi}_f(x'')\gamma^\nu\psi_i(x'') = e\cdot\overline{u}(p_4)\gamma^\nu u(p_2)e^{i(p_4-p_2)x''}$$

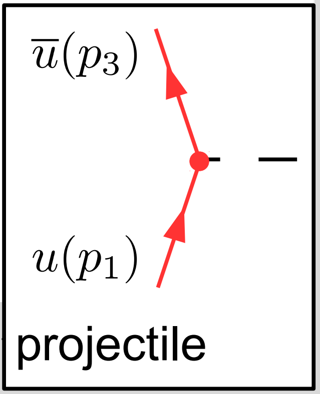


- Combination with photon propagator to get the evolution of A_μ:

$$\begin{aligned}A_\mu(x') &= e\int \mathrm{d}^4x''D^{\mu\nu}(x'-x'')J^\nu(x'') \\ &= e\cdot\int \mathrm{d}^4x''(2\pi)^{-4}\int \mathrm{d}^4q\,\frac{-g_{\mu\nu}}{q^2+i\epsilon}e^{i(p_4-p_2+q)x''}e^{-iqx'}\overline{u}(p_4)\gamma^\nu u(p_2) \\ &= e\cdot\int \mathrm{d}^4q\,\frac{-g_{\mu\nu}}{q^2+i\epsilon}\delta^4(p_4-p_2+q)e^{-iqx'}\overline{u}(p_4)\gamma^\nu u(p_2)\end{aligned}$$

- *Ansatz* for projectile current:

$$\overline{\phi}_f(x') = \overline{u}(p_3)e^{ip_3x'} \qquad \phi_i(x') = u(p_1)e^{-ip_1x'}$$



On the way to completion...

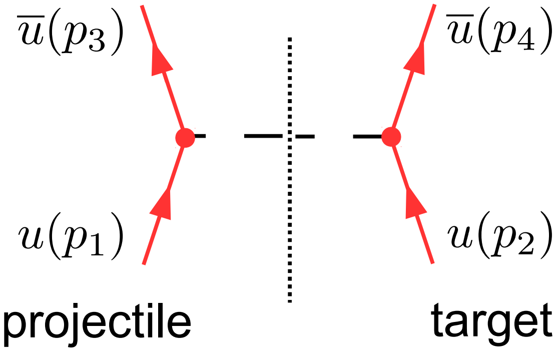
- 1^{st} order matrix element:

$$\mathcal{S}_{fi}^{(1)} = i \cdot \int \mathrm{d}^4x' e\, \overline{\phi}_f(x') \gamma^\mu A_\mu(x') \phi_i(x')$$

$$\overline{\phi}_f(x') = \overline{u}(p_3) e^{ip_3x'}$$

$$\phi_i(x') = u(p_1) e^{-ip_1x'}$$

$$A_\mu(x') = e \cdot \int \mathrm{d}^4q \tfrac{-g_{\mu\nu}}{q^2+i\epsilon} \delta^4(p_4-p_2+q) e^{-iqx'} u(p_4)\gamma^\nu u(p_2)$$



$$\mathcal{S}_{fi}^{(1)} = ie^2 \cdot \int \mathrm{d}^4q \underbrace{\int \mathrm{d}^4x'\; e^{i(p_3-p_1-q)x'}}_{(2\pi)^4\delta^4(p_3-p_1-q)} \overline{u}(p_3)\gamma^\mu u(p_1) \frac{-g_{\mu\nu}}{q^2+i\epsilon} \delta^4(p_4-p_2+q) \overline{u}(p_4)\gamma^\nu u(p_2)$$

$$= i\left((2\pi)^2 e\right)^2 \cdot \int \mathrm{d}^4q\; \delta^4(p_3-p_1-q) \overline{u}(p_3)\gamma^\mu u(p_1) \frac{-g_{\mu\nu}}{q^2+i\epsilon} \delta^4(p_4-p_2+q) \overline{u}(p_4)\gamma^\nu u(p_2)$$