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Teilchenphysik 2 — W/Z/Higgs an Collidern

Sommersemester 2019

Exercises No. 1

Discussion on May 8, 2019

Exercise 1: Higgs-Boson Production at Hadron Colliders

Consider on-shell Higgs-boson production at hadron colliders. Assuming the Higgs boson is produced at rest in the laboratory frame, what is the rapidity of the Higgs boson? Determine the momentum fraction x of the initial partons (quarks or gluons) in the production process at the Tevatron and the LHC.

Which parton has the highest probability at this x? You can draw the parton distribution functions (PDFs) using the applet at `http://hepdata.cedar.ac.uk/pdf/pdf3.html`. Which value of the momentum transfer Q^2 do you have to use?

From this, can you motivate the dominant Higgs-boson production channel?

Without the assumption that the Higgs boson is produced at rest, what is the minimal value of x to produce a Higgs boson at the Tevatron and the LHC?

Exercise 2: Equations of Motion

The equations of motion of a system described by the field $\Phi(x)$ can be derived from the Lagrange density $\mathcal{L}$ using the Euler-Lagrange equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi(x))} - \frac{\partial \mathcal{L}}{\partial \Phi(x)} = 0\,. \tag{1}$$

Show that the Lagrange density

$$\mathcal{L} = \overline{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi \tag{2}$$

describes a fermion field ψ, by deriving the Dirac equation using Eq. (1). Perfom the calcuation for both $\Phi = \overline{\psi}$ and $\Phi = \psi$.

Analogously, derive the equations of motion for a complex scalar field $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ from the Lagrange density

$$\mathcal{L} = \tfrac{1}{2}\left[(\partial_\mu\phi)\,(\partial^\mu\phi)^* - m^2\phi^2\right] .$$

What is the interpretation of the two obtained equations of motion?

Exercise 3: QED Gauge Field

Invariance of the Lagrangian Eq. (2) of a free fermion ψ field under local U(1) phase transformations $\mathrm{e}^{i\alpha(x)}$ can be achieved by replacing the partial derivative with the covariant derivative

$$\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu \,. \tag{3}$$

The gauge field A_μ can be interpreted as the photon of QED, the coupling constant e as the electric charge. As a consequence, the fermion is no longer free but interacts with the photon field.

In order to achieve local gauge invariance, the covariant derivative is required to transform as

$$D_\mu \to D'_\mu = D_\mu - i\partial_\mu\alpha(x) \,.$$

Show explicitly why this particular transformation behaviour is required. What is the required transformation behaviour of the gauge field A_μ?

For a consistent theory of QED, a kinetic term $\mathcal{L}_{\text{kin}}$ for the gauge field needs to be added to Eq. (2) in addition to the replacement Eq. (3). In the lecture, we have used

$$\mathcal{L}_{\text{kin}} = \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,.$$

Using the transformation behaviour of the gauge field A_μ obtained above, prove that the field-strength tensor $F_{\mu\nu}$ is locally gauge invariant, i. e. that

$$F'_{\mu\nu} = F_{\mu\nu} \,.$$

As a consequence, $\mathcal{L}_{\text{kin}}$ is also gauge invariant.

In order to further investigate the association of the gauge field A_μ with the QED photon, derive the equations of motion of A_μ by applying the Euler-Lagrange equations (1) to $\mathcal{L}_{\text{kin}}$. Show that this leads to the Proca equation when using the Lorenz gauge $\partial_\mu A^\mu = 0$ of electrodynamics.

Bonus: Show that the field-strength tensor can be conveniently written as

$$F_{\mu\nu} = -\tfrac{i}{e}[D_\mu, D_\nu] \,.$$

Exercise 4: Chiral Symmetry

The transformation $\chi : \psi \to \gamma^5\psi$ is called *chiral* transformation. It turns e. g. axial vectors into vectors and vice versa.

a) What is the adjoint of the transformed spinor?

b) Show that e_L and e_R are *eigentstates* of the chiral transformation with the *eigenvalues* -1 and $+1$, respectively.

c) Show that terms of type $\overline{\psi}\gamma^\mu\partial_\mu\psi$ are covariant under chiral transformations, while terms of type $\overline{\psi}m\psi$ are not. As a consequence the presence of light particles is a small perturbation of a chiral symmetry in the SM Lagrangian density.

Solutions

Useful definitions:

$$A^\dagger \equiv (A^*)^T\,, \qquad \overline{A} \equiv A^\dagger \gamma^0\,.$$

Useful identities of γ^μ matrices:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,, \qquad \gamma^0 = (\gamma^0)^\dagger\,, \qquad \gamma^a = -(\gamma^a)^\dagger\,,$$

and of γ^5:

$$\gamma^5 = (\gamma^5)^\dagger\,, \qquad (\gamma^5)^2 = 1\,, \qquad \{\gamma^5, \gamma^\mu\} = 0\,.$$

It is further

$$(A \cdot B)^\dagger = B^\dagger \cdot A^\dagger\,.$$

Solution to Exercise 1

The rapidity is a measure of the velocity in z direction:

$$y = \frac{1}{2} \ln \frac{E + p_z}{E - p_z}\,.$$

For a Higgs boson at rest in the laboratory frame, it is $p_z = 0$ and thus $y = 0$. In this case, the momentum fractions x_1 and x_2 of the two initial-state partons have to be the same.

More formally, the four-momenta of the colliding partons (relativistic regime $m \ll E$) in the laboratory frame are given by

$$p_1 = \frac{\sqrt{s}}{2}(x_1, 0, 0, x_1)\,, \qquad p_2 = \frac{\sqrt{s}}{2}(x_2, 0, 0, -x_2)\,,$$

and the rapidity can be written as

$$y = \frac{1}{2} \ln \frac{E + p_z}{E - p_z} = \frac{1}{2} \ln \frac{x_1 + x_2 + (x_1 - x_2)}{x_1 + x_2 - (x_1 - x_2)} = \frac{1}{2} \ln \frac{x_1}{x_2}\,,$$

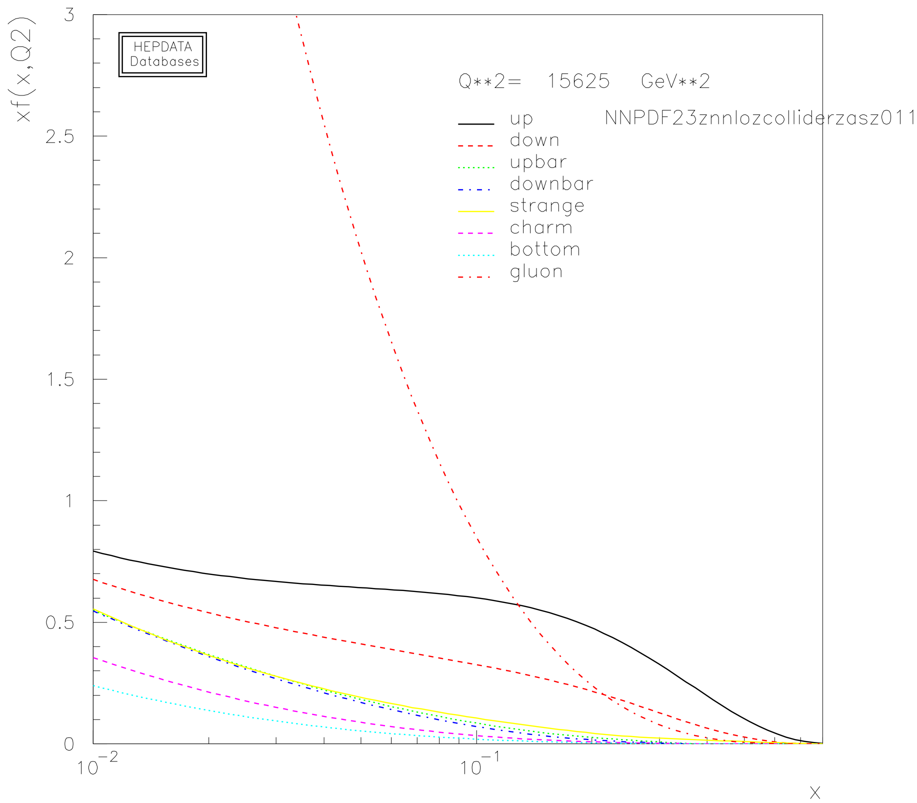
and for $y = 0$ follows $x_1 = x_2 \equiv x$. The invariant mass squared of the final state is

$$\hat{s} = x_1 x_2 s = x^2 s \stackrel{!}{=} m_H^2 \qquad \text{such that} \qquad x = \frac{m_H}{\sqrt{s}}\,.$$

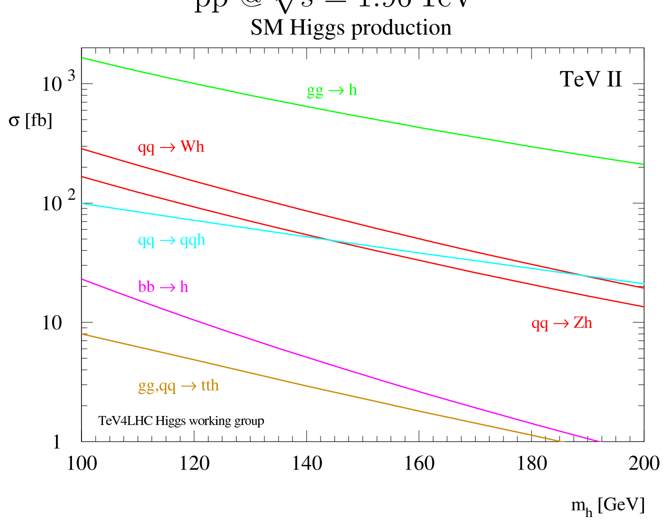
Thus, assuming $m_H = 125\,\mathrm{GeV}$, it is $x(y = 0)$ as in the following table:

accelerator	$\sqrt{s}\,[\mathrm{TeV}]$	$x(y = 0)$	x_{min}
Tevatron	1.96	0.064	$4.07 \cdot 10^{-3}$
LHC	7	0.018	$0.32 \cdot 10^{-3}$
LHC	8	0.016	$0.24 \cdot 10^{-3}$
LHC	13	0.010	$0.09 \cdot 10^{-3}$

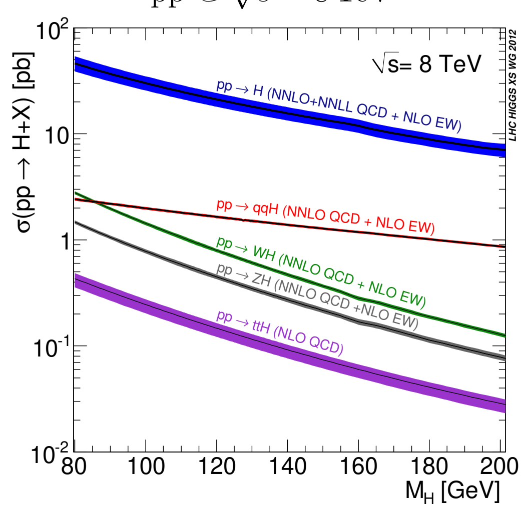
Using $Q^2 = m_H^2 = (125\,\mathrm{GeV})^2$ and for example the NNPDF set version 23, one finds the following parton distributions:



$p\overline{p}$ @ $\sqrt{s} = 1.96\,\mathrm{TeV}$



pp @ $\sqrt{s} = 8\,\mathrm{TeV}$



In general, it is

$$\hat{s} = x_1 x_2 s \stackrel{!}{=} m_H^2 \quad \rightarrow \quad x_2 = \frac{m_H^2}{s x_1}\,.$$

Since $x_1 \leq 1$, it is

$$x_2 > \frac{m_H^2}{s}$$

(and likewise for x_1). Thus, assuming $m_H = 125\,\mathrm{GeV}$, the minimal momentum fraction x_{min} is as given in the above table.

Solution to Exercise 2

Lagrangian for a free fermion of mass m,

$$\mathcal{L} = \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi = i\overline{\psi}\gamma^\mu\partial_\mu\psi - \overline{\psi}m\psi\ ,$$

in Euler-Lagrange equations, first for $\overline{\psi}$,

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})} - \frac{\partial\mathcal{L}}{\partial\overline{\psi}} = 0$$

$$\Rightarrow \quad \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})} = 0$$
$$\frac{\partial\mathcal{L}}{\partial\overline{\psi}} = i\gamma^\mu\partial_\mu\psi - m\psi$$

leads to

$$\begin{aligned} -i\gamma^\mu\partial_\mu\psi + m\psi &= 0 \\ \left(i\gamma^\mu\partial_\mu - m\right)\psi &= 0\,. \end{aligned}$$

The Euler-Lagrange equations for ψ are

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} - \frac{\partial\mathcal{L}}{\partial\psi} = 0$$

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = \partial_\mu\left(\overline{\psi}i\gamma^\mu\right) = i\partial_\mu\overline{\psi}\gamma^\mu$$
$$\frac{\partial\mathcal{L}}{\partial\psi} = -\overline{\psi}m\,,$$

which leads to

$$\begin{aligned} i\partial_\mu\overline{\psi}\gamma^\mu + \overline{\psi}m &= 0 \\ \overline{\psi}\left(i\gamma^\mu\partial_\mu + m\right) &= 0\,. \end{aligned} \tag{4}$$

This is the usual formulation of the adjoint *Dirac* equation. But a) the original *Dirac* equation contains a "$-$" and not a "$+$" in the braces and b) the derivative operator acts to the right and not to the left as suggested by this notation.

Definition of $\overline{\psi}$

$$\overline{\psi} = \psi^\dagger \gamma^0$$

in equation (4):

$$\begin{aligned}
i\partial_\mu \overline{\psi} \gamma^\mu + \overline{\psi} m &= 0 \qquad | \qquad \overline{\psi} = \psi^\dagger \gamma^0 \\
i\partial_\mu \psi^\dagger \gamma^0 \gamma^\mu + \psi^\dagger \gamma^0 m &= 0 \qquad | \qquad (\dots)^\dagger \\
\left(i\partial_\mu \psi^\dagger \gamma^0 \gamma^\mu + \psi^\dagger \gamma^0 m \right)^\dagger &= 0 \\
-i\gamma^{\mu\dagger} \gamma^{0\dagger} \partial_\mu \psi + m\gamma^{0\dagger} \psi &= 0 \\
-i\gamma^{\mu\dagger} \gamma^0 \partial_\mu \psi + m\gamma^0 \psi &= 0 \qquad | \qquad \gamma^0 \cdot \\
-i \underbrace{\gamma^0 \gamma^{\mu\dagger} \gamma^0}_{\equiv \gamma^\mu} \partial_\mu \psi + \underbrace{m\gamma^0 \gamma^0}_{\equiv 1} \psi &= 0 \qquad | \qquad \cdot (-1) \\
\left(i\gamma^\mu \partial_\mu - m \right) \psi &= 0 \,.
\end{aligned}$$

It is $\gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu$ because

$$\begin{aligned}
\gamma^0 \gamma^{0\dagger} \gamma^0 &= \gamma^0 \gamma^0 \gamma^0 = (\underbrace{\gamma^0 \gamma^0}_{=1}) \gamma^0 = \gamma^0 \\
\gamma^0 \gamma^{a\dagger} \gamma^0 &= \gamma^0 (-\gamma^a) \gamma^0 = \gamma^0 \gamma^0 \gamma^a = \gamma^a \,.
\end{aligned}$$

Applying the Euler-Lagrange equations to

$$\mathcal{L} = \tfrac{1}{2} \left[(\partial_\mu \phi) (\partial^\mu \phi)^* - m^2 \phi \phi^* \right]$$

yields

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\begin{aligned}
\Rightarrow \qquad \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} &= \partial_\mu \left(\tfrac{1}{2} \partial^\mu \phi^* \right) = \tfrac{1}{2} \partial_\mu \partial^\mu \phi^* \\
\frac{\partial \mathcal{L}}{\partial \phi} &= -\tfrac{1}{2} m^2 \phi^* \,,
\end{aligned}$$

resulting in

$$\left(\partial_\mu \partial^\mu + m^2 \right) \phi^* = 0 \,.$$

This is the Klein-Gordon equation for a scalar particle with mass m. Analogously, one obtains

$$\left(\partial_\mu \partial^\mu + m^2 \right) \phi = 0$$

when applying the Euler-Lagrange equations for ϕ.

The fields ϕ and ϕ^* can be interpreted to describe particles with the same mass but opposite charge. The effect of an external field A_μ can be inspected by rewriting the Lagrangian with the covariant derivative

$$\begin{aligned}
\mathcal{L} &= D_\mu\phi\,(D^\mu\phi)^* - m^2\phi\phi^* \\
&= (\partial_\mu\phi + ieA_\mu\phi)(\partial^\mu\phi^* - ieA^{\mu*}\phi^*) - m^2\phi\phi^* \\
&= \partial_\mu\phi\partial^\mu\phi^* + ieA_\mu\partial^\mu\phi^*\phi - ieA^{\mu*}\partial_\mu\phi\phi^* + e^2A_\mu A^{\mu*}\phi\phi^* - m^2\phi\phi^* \\
&= \underbrace{\partial_\mu\phi\partial^\mu\phi^* - m/2\phi\phi^*}_{\text{free boson part}} + \underbrace{ie\phi A_\mu\partial^\mu\phi^* - ie\phi^* A^{\mu*}\partial_\mu\phi + e^2A_\mu A^{\mu*}\phi\phi^*}_{\text{interaction part}}
\end{aligned}$$

The interaction terms have opposite sign in the charge e for ϕ and ϕ^*!

Solution to Exercise 3

The solution is motivated using the Lagrangian of a fermion field ψ as example. The Lagrangian for the U(1)-transformed case is, using already the covariant derivative with the given transformation behaviour,

$$\begin{aligned}
\mathcal{L}' &= \overline{\psi}'(i\gamma^\mu D'_\mu - m)\psi' \\
&= \overline{\psi}\mathrm{e}^{-i\alpha(x)}\left(i\gamma^\mu\left(D_\mu - i(\partial_\mu\alpha(x))\right) - m\right)\mathrm{e}^{i\alpha(x)}\psi \\
&= \overline{\psi}\mathrm{e}^{-i\alpha(x)}\left(i\gamma^\mu\left(\partial_\mu + ieA_\mu - i(\partial_\mu\alpha(x))\right) - m\right)\mathrm{e}^{i\alpha(x)}\psi \\
&= \overline{\psi}\left(i\gamma^\mu\left(\partial_\mu + i(\partial_\mu\alpha(x)) + ieA_\mu - i(\partial_\mu\alpha(x))\right) - m\right)\psi \\
&= \overline{\psi}\left(i\gamma^\mu\left(\partial_\mu + ieA_\mu\right) - m\right)\psi \\
&= \overline{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi = \mathcal{L}\,.
\end{aligned}$$

The crucial step is

$$\begin{aligned}
\partial_\mu\mathrm{e}^{i\alpha(x)}\psi &= \mathrm{e}^{i\alpha(x)}(\partial_\mu\psi) + (\partial_\mu\mathrm{e}^{i\alpha(x)})\psi \\
&= \mathrm{e}^{i\alpha(x)}(\partial_\mu\psi) + \mathrm{e}^{i\alpha(x)}(\partial_\mu i\alpha(x))\psi \\
&= \mathrm{e}^{i\alpha(x)}\left(\partial_\mu\psi + i(\partial_\mu\alpha(x))\psi\right) \\
&= \mathrm{e}^{i\alpha(x)}\left(\partial_\mu + i(\partial_\mu\alpha(x))\right)\psi
\end{aligned}$$

The transformation of the covariant derivative resulted in

$$\begin{aligned}
D'_\mu &= D_\mu - i\partial_\mu\alpha(x) \\
&= \partial_\mu + ieA_\mu - i\partial_\mu\alpha(x)\,.
\end{aligned}$$

On the other hand, gauge invariance means that $D'_\mu = \partial_\mu + ieA'_\mu$, and thus

$$D'_\mu = \partial_\mu + ieA_\mu - i\partial_\mu\alpha(x) = \partial_\mu + ieA'_\mu\,,$$

which results in

$$A'_\mu = A_\mu - \tfrac{1}{e}\partial_\mu\alpha\,.$$

Transforming

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

yields

$$\begin{aligned}
F'_{\mu\nu} &= \partial_\mu A'_\nu - \partial_\nu A'_\mu \\
&= \partial_\mu (A_\nu - \frac{1}{e}\partial_\nu \alpha) - \partial_\nu (A_\mu - \frac{1}{e}\partial_\mu \alpha) \\
&= \partial_\mu A_\nu - \frac{1}{e}\partial_\mu \partial_\nu \alpha - \partial_\nu A_\mu + \frac{1}{e}\partial_\nu \partial_\mu \alpha \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}\,,
\end{aligned}$$

i. e. $F_{\mu\nu}$ is local gauge invariant.

In order to derive the equations of motion for A_μ from

$$\mathcal{L}_{\text{kin}} = \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,,$$

it is convenient to rewrite

$$\begin{aligned}
F_{\mu\nu} F^{\mu\nu} &= (\partial_\mu A_\nu - \partial_\nu A_\mu)\,(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= 2\,(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu) \\
&= 2\,(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu) \\
&= 2\left(g_{\mu\alpha} g_{\nu\beta} \partial^\alpha A^\beta \partial^\mu A^\nu - g_{\mu\beta} g_{\nu\alpha} \partial^\alpha A^\beta \partial^\mu A^\nu\right) \\
&= 2\,(g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha})\,(\partial^\alpha A^\beta)(\partial^\mu A^\nu)\,.
\end{aligned}$$

Thus, one obtains for the equation of motions

$$\partial^\rho \frac{\partial \mathcal{L}}{\partial(\partial^\rho A^\sigma)} - \frac{\partial \mathcal{L}}{\partial A^\sigma} = 0$$

$$\begin{aligned}
\Rightarrow \quad \partial^\rho \partial \frac{\mathcal{L}}{\partial(\partial^\rho A^\sigma)} &= \partial^\rho \tfrac{1}{4} \left(2\,(g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha})\,\delta^\alpha_\rho \delta^\beta_\sigma \partial^\mu A^\nu + (\alpha \leftrightarrow \mu, \beta \leftrightarrow \nu)\right) \\
&= \partial^\rho \tfrac{1}{4}\,(4\,(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})\,\partial^\mu A^\nu) \\
&= \partial^\rho\,(\partial_\rho A_\sigma - \partial_\sigma A_\rho) \\
&= \partial^\rho \partial_\rho A_\sigma - \partial_\sigma \underbrace{\partial^\rho A_\rho}_{=0}\,,
\end{aligned}$$

that is the Proca equation for a massless vector particle:

$$\partial^\nu \partial_\nu A_\mu = 0\,.$$

It is

$$
\begin{aligned}
-\tfrac{i}{e}\left[D_\mu, D_\nu\right] &= -\tfrac{i}{e}\left[\partial_\mu + ieA_\mu, \partial_\nu + ieA_\nu\right] \\
&= -\tfrac{i}{e}\big(\left[\partial_\mu, \partial_\nu\right] - e^2\left[A_\mu, A_\nu\right] + ie\left[\partial_\mu, A_\nu\right] + ie\left[A_\mu, \partial_\nu\right]\big) \\
&= -\tfrac{i}{e}\left(ie\left[\partial_\mu, A_\nu\right] + ie\left[A_\mu, \partial_\nu\right]\right) \\
&= \left[\partial_\mu, A_\nu\right] + \left[A_\mu, \partial_\nu\right] \\
&= \left[\partial_\mu, A_\nu\right] - \left[\partial_\nu, A_\mu\right] \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu\,,
\end{aligned}
$$

where in the last step the fact is used that $[\partial_\mu, A_\nu]$ has to be understood as an operator acting on a test function f, such that

$$
\begin{aligned}
[\partial_\mu, A_\nu]f &= \partial_\mu A_\nu f - A_\nu \partial_\mu f \\
&= (\partial_\mu A_\nu)f + A_\nu(\partial_\mu f) - A_\nu \partial_\mu f \\
&= (\partial_\mu A_\nu)f\,.
\end{aligned}
$$

Solution to Exercise 4

a)

$$
\overline{\psi} \to (\gamma^5\psi)^\dagger\gamma^0 = \psi^\dagger\gamma^5\gamma^0 = -\overline{\psi}\gamma^5
$$

b)

$$
\begin{aligned}
\psi_L \to \gamma^5\psi_L = \gamma^5\frac{1}{2}(1-\gamma^5)\psi = -\psi_L \\
\psi_R \to \gamma^5\psi_R = \gamma^5\frac{1}{2}(1+\gamma^5)\psi = +\psi_R
\end{aligned}
$$

c)

$$
\overline{\psi}\gamma^\mu\partial_\mu\psi \to (-\overline{\psi}\gamma^5)\gamma^\mu\partial_\mu(\gamma^5\psi) = +\overline{\psi}\gamma_\mu\gamma^5\gamma^5\partial_\mu\psi = \overline{\psi}\gamma_\mu\partial_\mu\psi
$$

$$
\overline{\psi}m\partial_\mu\psi \to (-\overline{\psi}\gamma^5)m(\gamma^5\psi) = -\overline{\psi}m\psi
$$