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Teilchenphysik 2 — W/Z/Higgs an Collidern

Sommersemester 2019

Specialisations No. 1

Discussion on May 28, 2019

Exercise 1: Quark Mixing

In the most general case, the Yukawa coupling terms for quarks can be written as

$$\mathcal{L}_{\text{Yukawa}}^{\text{quarks}} = -G^{d}_{ij}\overline{Q}'_{L,i}\phi d'_{R,j} - G^{u}_{i,j}\bar{Q}'_{L,i}\phi^{c}u'_{R,j} - h.c.$$

with complex matrices G_{ij}, where the i, j denote the generation, and with the Q'_L denoting an $\mathrm{SU}(2)_L$ iso-spin doublet of quarks, e. g. $(u \;\; d)'$, and the q'_R denoting the corresponding singlets, e. g. u'_R, d'_R. The quark states are written in the SU(2)-interaction base as denoted by the prime. The corresponding electroweak Lagrangian can be written in the same notation as

$$\mathcal{L}_{\text{EWK}} = i\overline{Q}'_L\gamma^{\mu}\left[\partial_{\mu} + i\tfrac{g}{2}\mathrm{W}^{a}_{\mu}\tau^{a} + i\tfrac{g'}{2}Y_L\mathrm{B}_{\mu}\right]Q'_L + i\overline{q'}_R\gamma^{\mu}\left[\partial_{\mu} + i\tfrac{g'}{2}Y_R\mathrm{B}_{\mu}\right]q'_R\,,$$

where only the terms involving fermions, i. e. the covariant derivatives, are considered here. The τ^a, $a = 1, 2, 3$, represent the generators of the SU(2) gauge group and can be written as the Pauli matrices $\tau^a = \sigma^a$.

Show that the charged-current interaction ($\mathrm{W}^{\pm}$ boson exchange) part follows to

$$\mathcal{L}_{\text{CC}} = \overline{Q}'_L\gamma^{\mu}\mathrm{W}^{\pm}_{\mu}\tau^{\pm}Q'_L = \overline{(u \;\; d)}'_{L,i}\gamma^{\mu}\mathrm{W}^{\pm}_{\mu}\tau^{\pm}\begin{pmatrix}u\\ d\end{pmatrix}'_{L,i}, \tag{1}$$

where $u_{L,i}$ and $d_{L,i}$ denote the up- and down-type quarks of generation i, the $\mathrm{W}^{\pm}_{\mu}$ fields are defined as $\mathrm{W}^{\pm}_{\mu} = \frac{1}{\sqrt{2}}(\mathrm{W}^{1}_{\mu} \mp i\mathrm{W}^{2}_{\mu})$, and the $\tau^{\pm} = \frac{1}{2}(\tau^1 \pm i\tau^2)$ are the ladder operators in SU(2) iso-spin space.

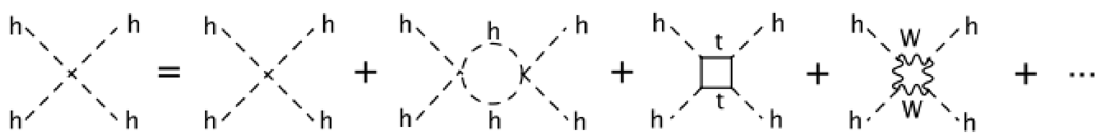
Equation (1) is still written in the SU(2) interaction base, as indicated by the primed spinors. Express $\mathcal{L}_{\text{CC}}$ in terms of the quark mass eigenstates $u_{L,i}$ etc. and show that the $W^\pm$ bosons can be exchanged between quarks of different families, i. e. that quark mixing appears in the charged-current interaction.

Why does quark mixing not occur for the neutral current interactions?

<u>Exercise 2</u>: Theoretical Bounds on the Higgs-Boson Mass

Even though the Higgs-boson mass is not predicted within the Standard Model, upper and lower bounds can be derived from internal consistency considerations.

Investigate the running of the Higgs self-coupling parameter $\lambda = \lambda(Q^2)$ with the energy scale Q^2. The running is driven by contributions from Higgs-boson, top-quark, and gauge-boson loops as depicted below:



In the following, consider only the Higgs-boson and top-quark contributions and neglect gauge-boson contributions. Study the two limits of very large and very small Higgs-boson masses $m_{\mathrm{H}}^2 = 2\lambda(Q^2 = v^2)v^2$ at the electroweak scale $Q = v = 246\,\mathrm{GeV}$. The two limits may correspond $\lambda \gg y_t$ and $\lambda \ll y_t$, respectively, where y_t denotes the top-Higgs Yukawa coupling constant. Use the approximate solutions to the one-loop renormalisation group equation in these two cases as given in the lecture to relate the value of λ at the electroweak scale to its value at a higher scale $Q = \Lambda \gg v$.

Requiring that the Higgs self-coupling remains finite at all scales Q up to Λ, i. e. requiring that $\lambda(\Lambda^2) < \infty$, derive an upper limit for $\lambda(v^2)$ at the electroweak scale (*triviality bound*). Similarly, requiring that the Higgs potential develops a defined minimum for all scales up to Λ, derive a lower limit for $\lambda(v^2)$ (*stability bound*).

Use the results to compute bounds on the Higgs-boson mass $m_{\mathrm{H}}^2 = 2\lambda(v^2)v^2$ at the electroweak scale, depending on the scale Λ up to which the Standard Model is extrapolated.

Calculate the bounds on m_H explicitly for $\Lambda = 10\,\mathrm{TeV}$ and $\Lambda = 10^{19}\,\mathrm{GeV}$. What do you notice?

Solutions

Useful definitions:

$$A^\dagger \equiv (A^*)^T\,, \qquad \overline{A} \equiv A^\dagger \gamma^0\,.$$

Useful identities of γ^μ matrices:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,, \qquad \gamma^0 = (\gamma^0)^\dagger\,, \qquad \gamma^a = -(\gamma^a)^\dagger\,,$$

and of γ^5:

$$\gamma^5 = (\gamma^5)^\dagger\,, \qquad (\gamma^5)^2 = 1\,, \qquad \{\gamma^5, \gamma^\mu\} = 0\,.$$

It is further

$$(A \cdot B)^\dagger = B^\dagger \cdot A^\dagger\,.$$

Solution to Exercise 1

The charged-current interaction follows from the terms involving the W^1_μ and W^2_μ fields (see lecture 2), i.e.

$$\mathcal{L}_{\mathrm{CC}} = -\tfrac{g}{2}\overline{Q}'_L \gamma^\mu (\mathrm{W}^1_\mu \tau^1 + \mathrm{W}^2_\mu \tau^2) Q'_L\,,$$

and with

$$\mathrm{W}^1_\mu \tau^1 + \mathrm{W}^2_\mu \tau^2 = \sqrt{2}\left(\mathrm{W}^+_\mu \tau^+ + \mathrm{W}^-_\mu \tau^-\right) \qquad \text{(see lecture 2)}$$

one obtains

$$\mathcal{L}_{\mathrm{CC}} = -\tfrac{g}{\sqrt{2}}\overline{Q}'_L \gamma^\mu \left[\mathrm{W}^+_\mu \tau^+ + \mathrm{W}^-_\mu \tau^-\right] Q'_L = -\tfrac{g}{\sqrt{2}}\overline{(u\ \ d)}'_{L,i} \gamma^\mu \left[\mathrm{W}^+_\mu \tau^+ + \mathrm{W}^-_\mu \tau^-\right] \begin{pmatrix} u \\ d \end{pmatrix}'_{L,i}\,.$$

With

$$\tau^+ = \tfrac{1}{2}(\tau^1 + i\tau^2) = \tfrac{1}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

follows for the W^+ case

$$\begin{aligned}
\mathcal{L}_{\mathrm{CC}+} &= -\tfrac{g}{\sqrt{2}}\overline{(u\ \ d)}'_{L,i} \gamma^\mu \mathrm{W}^+_\mu \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}'_{L,i} \\
&= -\tfrac{g}{\sqrt{2}}\overline{(u\ \ d)}'_{L,i} \gamma^\mu \mathrm{W}^+_\mu \begin{pmatrix} d \\ 0 \end{pmatrix}'_{L,i} \\
&= -\tfrac{g}{\sqrt{2}}\overline{u}'_{L,i} \gamma^\mu \mathrm{W}^+_\mu d'_{L,i} \\
&= -\tfrac{g}{\sqrt{2}}\gamma^\mu \mathrm{W}^+_\mu \overline{u}'_{L,i} d'_{L,i} \\
&= -\tfrac{g}{\sqrt{2}}\gamma^\mu \mathrm{W}^+_\mu \overline{u}_{L,i} \underbrace{(V^u_L V^{d\dagger}_L)_{ij}}_{V_{\mathrm{CKM},ij}} d_{L,j}
\end{aligned}$$

because with the definition of the mass eigenstates

$$\begin{aligned} d_{L,i} &= (V_L^d)_{ij} d'_{L,j} \\ u_{L,i} &= (V_L^u)_{ij} u'_{L,j} \end{aligned}$$

follows

$$\begin{aligned} d_{L,i} &= \qquad (V_L^d)_{ij} d'_{L,j} \\ (V_L^d)^\dagger_{ij} d_{L,i} &= (V_L^d)^\dagger_{ij} (V_L^d)_{ij} d'_{L,j} = d'_{L,j} \end{aligned}$$

and

$$\begin{aligned} u_{L,i} \qquad &= (V_L^u)_{ij} u'_{L,j} \\ \overline{u}_{L,i} \qquad &= \overline{(V_L^u)_{ij} u'_{L,j}} \\ &= (u'_{L,j})^\dagger (V_L^u)^\dagger_{ij} \gamma^0 \\ &= (u'_{L,j})^\dagger \gamma^0 (V_L^u)^\dagger_{ij} \\ &= \overline{u'}_{L,j} (V_L^u)^\dagger_{ij} \\ \overline{u}_{L,i} (V_L^u)_{ij} &= \overline{u'}_{L,j} (V_L^u)^\dagger_{ij} (V_L^u)_{ij} = \overline{u'}_{L,j} \end{aligned}$$

Analogously, one shows for W^-

$$\mathcal{L}_{\text{CC-}} = -\tfrac{g}{\sqrt{2}} \gamma^\mu \mathrm{W}^-_\mu \overline{d}_{L,i} (V_L^d V_L^{u\dagger})_{ij} u_{L,j} \, .$$

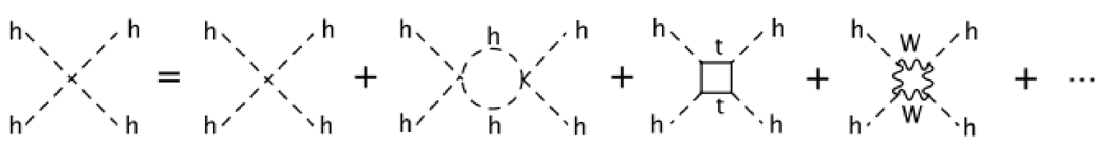
For the Z boson exchange and the kinetic terms, there is not mixing. Since there is no $\tau^\pm$ matrix involved, the quark bilinear forms consist of quarks of the same isospin only. Hence, the involved mixing matrix elements are of the form $V_L^d V_L^{d\dagger} = 1$.

Solution to Exercise 2

The running of the Higgs self-coupling constant λ is given by the renormalisation group equation

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\ln Q^2} = \beta = \frac{3}{4\pi^2} \Big[\underbrace{\lambda^2}_{\text{Higgs}} + \underbrace{\tfrac{1}{2}\lambda y_t^2 - \tfrac{1}{4} y_t^4}_{\text{top quark}} - \underbrace{\tfrac{1}{8}\lambda(3g^2 + g'^2)}_{\mathrm{W}^\pm,\,\mathrm{Z}\text{ bosons}} + \ldots \Big] \, , \tag{2}$$

given here at lowest order (one loop). It is driven by loop contributions from Higgs bosons, top-quarks, and gauge bosons as depicted below:



The value of λ at the electroweak scale v determines the value of the measured Higgs-boson mass

$$m_H^2 = 2\lambda v^2 = 2\lambda(v)v^2 \, ,$$

and hence, by investigating the running of λ, bounds on the Higgs boson mass can be derived. For this, it is instructive to consider the limits of very small and very large m_H.

Large m_H corresponding to $\lambda \gg y_t, g, g'$ In this case, the loop contributions from Higgs bosons to the running of λ dominate and only the term $\propto \lambda^2$ in (2) is of relevance. Then, a solution to (2) is given by (see lecture)

$$\lambda(Q^2) = \frac{\lambda(v^2)}{1 - \frac{3}{4\pi^2}\lambda(v^2)\ln\left(\frac{Q^2}{v^2}\right)} \, , \tag{3}$$

which relates the value of λ at the EWK scale v to its value at a higher scale Q. Inspection of (3) reveals that λ increases as Q increases, until it hits a pole when the denominator becomes 0. Requiring λ to be finite, i. e. $\lambda < \infty$, for all scales up to $Q = \Lambda$ leads to the constraint

$$\frac{3}{4\pi^2}\lambda^2(v^2)\ln\left(\frac{\Lambda^2}{v^2}\right) < 1 \quad \Rightarrow \quad \lambda(v^2) < \frac{4\pi^2}{3}\frac{1}{\ln\left(\frac{\Lambda^2}{v^2}\right)} \, .$$

This translates into an upper limit on the Higgs boson mass at the EWK scale of

$$m_H = \sqrt{2\lambda(v^2)v^2} < \sqrt{\frac{8\pi^2 v^2}{3\ln\left(\frac{\Lambda^2}{v^2}\right)}} \, .$$

NB: alternatively, one can apply a stronger constraint by requiring the theory to remain perturbative, i. e. to require $\lambda < 1$ at all Q. This changes the numerical result slightly, but the arguments stay the same.

Small m_H corresponding to $\lambda \ll y_t, g, g'$ In this case, the loop contributions from the top quark to the running of λ dominate, and therefore, one can neglect all but the term $\propto y_t^4$ in (2). Then, a solution to (2) is given by (see lecture)

$$\lambda(Q^2) = \lambda(v^2) - \frac{3}{4\pi^2}\frac{m_t^4}{v^4}\ln\left(\frac{Q^2}{v^2}\right) \, , \tag{4}$$

which relates the value of λ at the EWK scale v to its value at a higher scale Q. Inspection of (4) reveals that λ decreases as Q increases. In order for the Higgs mechanism to work, i. e. in order for spontaneous symmetry breaking to occur, the Higgs potential

$$V(\phi) = \mu^2|\phi|^2 + \lambda|\phi|^4$$

needs to develop a stable minimum at a value $|\phi| \neq 0$. Thus, it is necessary that $\lambda > 0$ at all scales Q up to Λ. From this, it follows

$$\lambda(v^2) > \frac{3}{4\pi^2}\frac{m_t^4}{v^4}\ln\left(\frac{\Lambda^2}{v^2}\right) .$$

This translates into a lower limit on the Higgs boson mass at the EWK scale of

$$m_H = \sqrt{2\lambda(v^2)v^2} > \sqrt{\frac{3}{2\pi^2}\ln\left(\frac{\Lambda^2}{v^2}\right)\frac{m_t^4}{v^2}} .$$

Note the strong dependence on the top-quark mass!

Resulting boundaries The above approximations lead to the following boundaries at the example scales given in the exercise (all units in GeV):

Λ	10^4	10^{19}
m_H^{up}	463	144
m_H^{low}	130	419

This result does not make sense for the high-energy case where $m_H^{\text{low}} > m_H^{\text{up}}$! The reason is in the approximations that have been made to derive the boundaries. In particular for the lower boundary, the neglected W and Z boson contributions play an important role as they enter with a different sign than the top quark, even if at lower magnitude. Furthermore, the considered β function is only at one-loop level; important modifications of the limits arise from yet higher-order contributions. Still, the principle argument can be understood with this simplified example.