Karlsruher Institut für Technologie

Teilchenphysik 2 — W/Z/Higgs an Collidern

Sommersemester 2019

Matthias Schröder und Roger Wolf | Vorlesung 8

INSTITUT FÜR EXPERIMENTELLE TEILCHENPHYSIK (ETP)

Termine: Bisher

Date	Room	Type	Topic
Wed Apr 24.	Kl. HS B	LE 01	1. Organisation and introduction: particle physics at colliders + W/Z/H history
Tue Apr 30.	30.23 11/12	—	*no class*
Wed May 01.	Kl. HS B	—	*no class*
Tue May 07.	30.23 11/12	LE 02	2.1 Gauge theory & 2.2 The electroweak sector of the SM I
Wed May 08.	Kl. HS B	LE 03, EX 01	2.3 Discovery of the W and Z bosons & EX gauge theories
Tue May 14.	30.23 11/12	LE 04	2.4 The Higgs mechanism
Wed May 15.	Kl. HS B	EX 02	Exercise “SM Higgs mechanism”
Tue May 21.	30.23 11/12	—	*no class*
Wed May 22.	Kl. HS B	LE 05	2.5 The electroweak sector of the SM II (Higgs mechanism + Yukawa couplings)
Tue May 28.	30.23 11/12	SP 01	Specialisation of 2.4 and 2.5
Wed May 29.	Kl. HS B	LE 06	3.1 From theory to observables & 3.2 Reconstruction + analysis of exp. data
Tue Jun 04.	30.23 11/12	EX 03	Exercise “Trigger efficiency measurement”
Wed Jun 05.	Kl. HS B	LE 07	3.3 Measurements in particle physics (part 1)
Tue Jun 11.	30.23 11/12	EX 04	Exercise on statistical methods
Wed Jun 12.	Kl. HS B	LE 08	3.3 Measurements in particle physics (part 2)
Tue Jun 18.	30.23 11/12	SP 02	Specialisation “Limit setting”
Wed Jun 19.	Kl. HS B	SP 03	Specialisation “Unfolding”
Tue Jun 25.	30.23 11/12	LE 09	4.1 Determination of SM parameters
Wed Jun 26.	Kl. HS B	LE 10	4.2 Measurement and role of W/Z bosons at the LHC
Tue Jul 02.	30.23 11/12	EX 05	Paper seminar “Z pole measurements”
Wed Jul 03.	Kl. HS B	LE 11	4.3 Processes with several W/Z bosons
Tue Jul 09.	30.23 11/12	EX 06	Paper seminar Higgs
Wed Jul 10.	Kl. HS B	LE 12	5.1 Discovery and first measurements of the Higgs boson
Tue Jul 16.	30.23 11/12	EX 07	Exercise “Machine learning in physics analysis”
Wed Jul 17.	Kl. HS B	LE 13	5.2 Measurement of couplings and kinematic properties
Tue Jul 23.	30.23 11/12	EX 08	Presentations: results of ML challenge
Wed Jul 24.	Kl. HS B	LE 14	5.3 Search for Higgs physics beyond the SM & 5.4 Future Higgs physics

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Analysis Chain

Nature $\overset{?}{\longleftrightarrow}$ Theory

$\Downarrow$ (Nature) $\Downarrow$ (Theory)

Detector: data recording
calibrated digitised data
online selection (trigger)

MC simulation
physics process
detector signals

$\Downarrow$

Physics object **reconstruction**
Event **selection**

$\Downarrow$

Statistical analysis: **results**
Comparison with theory

3. From Theory to Experiment (and Back)

3.1 From theory to observables
- Cross-section calculation: basic picture
- Fermion propagator and perturbation theory
- Scattering matrix and Feynman rules

3.2 Reconstruction of experimental data
- Reminder: accelerators and particle detectors
- Trigger
- Reconstruction of physics objects

3.3 Measurements in particle physics
- Parameter estimation
- Hypothesis testing
- Search for new physics (exclusion limits)

3.4 Monte Carlo simulation

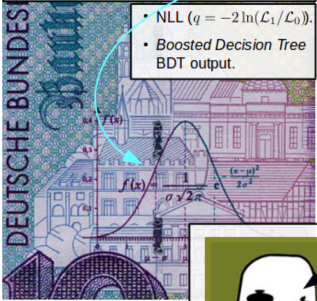
3.3 Measurements in particle physics

Statistics and Stochastic (Probability)

Test statistic:

$\Omega^n \to \mathbb{R}: \qquad x \to f(x)$

- NLL ($q = -2\ln(\mathcal{L}_1/\mathcal{L}_0)$).
- *Boosted Decision Tree* BDT output.



Probability (density) function:

$\Omega^n \to [0,1] \subset \mathbb{R}: \qquad x \to \mathcal{P}(x)$

- Problem of statistics is usually *ill-defined*.
- Deduce *truth* from shadows in Platon's cave...

From R. Wolf

Statistics and Particle Physics

Karlsruher Institut für Technologie

Experiment

- All measurements are derived from **counting experiments**
- **Millions of billions** of particle collision events each year
- Each event **independent** from and (to many extents) identical to all others

Theory

- Nature intrinsically stochastic
- QM wave functions interpreted as **probability density functions**
- Event-by-event simulation using **Monte Carlo** methods

Particle physics experiments perfect application of statistics

Literature

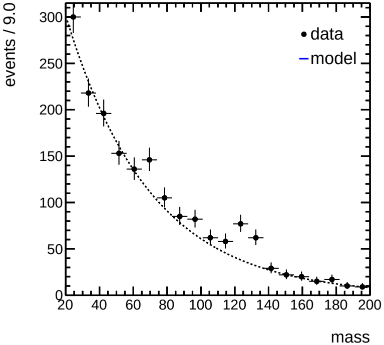
- R. J. Barlow, “Statistics: A Guide to the Use of Statistical Methods in the Physical Sciences”, The Manchester Physics Series **highly recommended**
- G. Cowan, “Statistical Data Analysis”, Oxford Science Publications
- Review “Statistics”, Chin. Phys. C, 40, 100001 (2016) **very good review**
- The ATLAS and CMS Collaborations, “Procedure for the LHC Higgs boson search combination in Summer 2011”, CMS-NOTE-2011-005, ATL-PHYS-PUB-2011-11 **expert document**

Likelihood

- **Likelihood** $\boxed{\mathcal{L}(n_{\text{obs}}; k) = \prod_i \mathcal{P}(n_{\text{obs},i}; k)}$ **quantifies compatibility of the observed data with a given model**
  - i: independent counting experiments, e. g. bins of a histogram
  - $n_{\text{obs}} = \{n_{\text{obs},i}\}$: number of observed events in i, distributed as $\mathcal{P}$
  - $k = \{k_j\}$: model parameters in $\mathcal{P}$

  → **function of model parameters and observed data**
- Example
  - Data: observed events in 20 bins
  - Model: known SM process (background)

$$\mathcal{L}(n_{\text{obs}}; b) = \prod_i \text{Poisson}(n_{\text{obs},i}, b)$$



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  - Data: observed events in 20 bins
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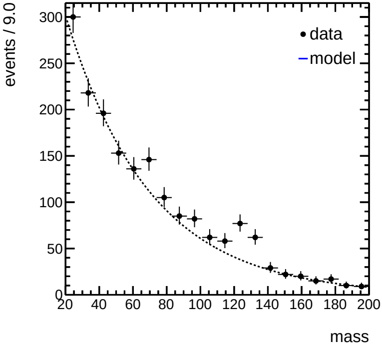
$$\mathcal{L}(n_{\text{obs}}; b) = \prod_i \frac{(n_{\text{pred},i})^{n_{\text{obs},i}}}{n_{\text{obs},i}!} \mathrm{e}^{-n_{\text{pred},i}}$$

$$n_{\text{pred},i}(b) = \underbrace{b \cdot \mathrm{e}^{-\alpha x_i}}_{\text{background}}$$



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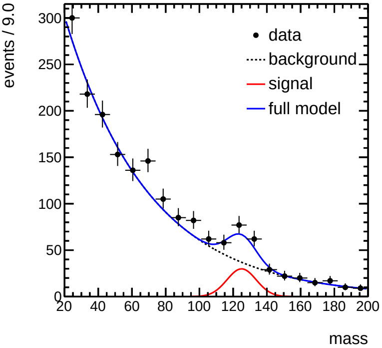
  - $k = \{k_j\}$: model parameters in $\mathcal{P}$
  - → **function of model parameters and observed data**
- Example
  - Data: observed events in 20 bins
  - Model: known SM process (background) + signal

$$\mathcal{L}(n_{\text{obs}}; b, s) = \prod_i \frac{(n_{\text{pred},i})^{n_{\text{obs},i}}}{n_{\text{obs},i}!} e^{-n_{\text{pred},i}}$$

$$n_{\text{pred},i}(b, s) = \underbrace{b \cdot e^{-\alpha x_i}}_{\text{background}} + \underbrace{s \cdot e^{-(m-x_i)^2}}_{\text{signal}}$$



Likelihood

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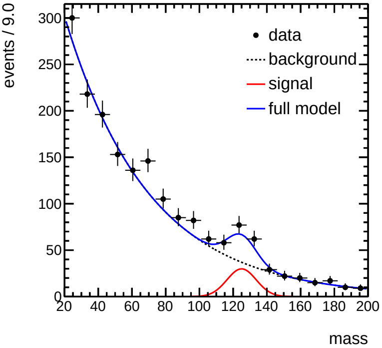
  $\rightarrow$ **function of model parameters and observed data**
- $\mathcal{L}$ is **not the probability** of a model!
  - Also not the probability of observing the data given the model
- $\mathcal{L}$ is the product of pdfs of the model, used to quantify the compatibility of the data with the model



3.3.1. Parameter estimation

Parameter Estimation

- Which parameters of a given model describe best the data?
- **Maximum likelihood (ML) method** ("maximum likelihood fit"): Which parameter values lead to the maximum of $\mathcal{L}$?

→ **Maximum likelihood estimators $\hat{\mathbf{k}}$** of the true values

$$\boxed{\max\left[\mathcal{L}(n_{\text{obs}}, k)\right] = \mathcal{L}(n_{\text{obs}}, \hat{k})}$$

- Technically: minimisation of negative log-likelihood

$$\text{NLL}(n_{\text{obs}}, k) = -2 \ln \mathcal{L}$$

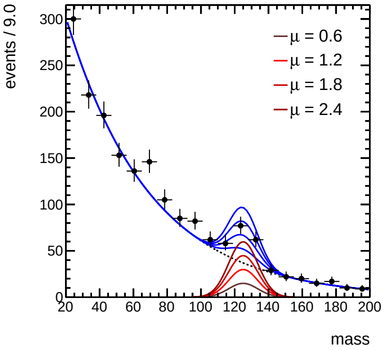
  - Logarithm is monotonic: retains minimum
  - Turns product of pdfs into sum: easier to minimise
- Arbitrarily complex problem, vast amount of techniques and literature
  - e. g. ATLAS+CMS Higgs couplings combination: $\approx$ 4250 parameters
  - e. g. CMS tracker-alignment problem: $\mathcal{O}(10^6)$ parameters

Maximum-Likelihood Method

- Example: which *signal-strength modifier* μ describes best the data?
  - Common in Higgs physics: $\mu = (\sigma \cdot \mathcal{B})/(\sigma_{\text{SM}} \cdot \mathcal{B}_{\text{SM}})$

$$n_{\text{pred},i}(\mu) = \underbrace{b \cdot e^{-\alpha x_i}}_{\text{background}} + \underbrace{\mu \cdot s_{\text{SM}} \cdot e^{-(m-x_i)^2}}_{\text{signal}}$$

$$\mathcal{L}(n_{\text{obs}};\mu) = \prod_i \frac{(n_{\text{pred},i})^{n_{\text{obs},i}}}{n_{\text{obs},i}!} e^{-n_{\text{pred},i}}$$



→ $\hat{\mu}$: maximises $\mathcal{L}(n_{\text{obs}};\mu)$

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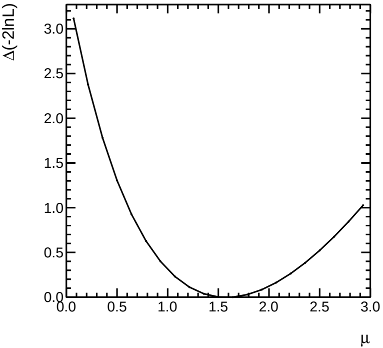


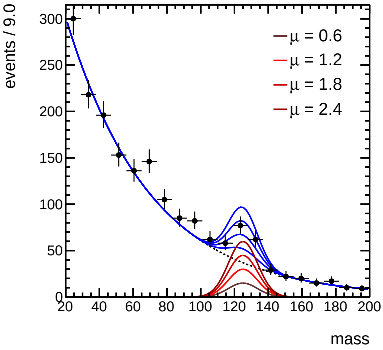
→ $\hat{\mu}$: minimises $-2 \ln \mathcal{L}(n_{\text{obs}};\mu)$

Maximum-Likelihood Method

- Example: which *signal-strength modifier* μ describes best the data?
  - Common in Higgs physics: $\mu = (\sigma \cdot \mathcal{B})/(\sigma_{\text{SM}} \cdot \mathcal{B}_{\text{SM}})$
- Important case: **Gaussian distributed measurements**
  - e. g. approximation of Poisson for large number of events (in practice > 10)
- NLL becomes

$$-2 \ln \mathcal{L} = \sum_i \frac{(n_{\text{obs},i} - n_{\text{pred},i})^2}{n_{\text{pred},i}} + \text{const}$$

$$\chi^2 \equiv \sum_i \frac{(n_{\text{obs},i} - n_{\text{pred},i})^2}{n_{\text{pred},i}}$$



Maximum-Likelihood Method

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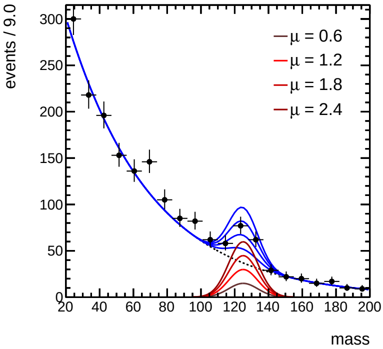
  - Common in Higgs physics: $\mu = (\sigma \cdot \mathcal{B})/(\sigma_{\text{SM}} \cdot \mathcal{B}_{\text{SM}})$

ML method obtains parameter values most compatible with the data *for a given model* — **a ML fit does not find the correct model**!

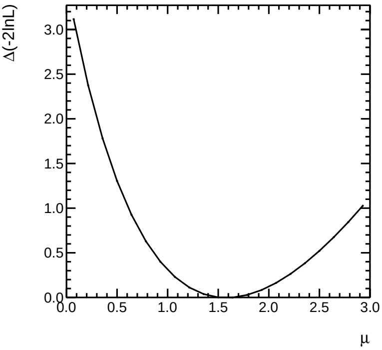
ML estimator is function of the observed data



Maximum-Likelihood Method

- Uncertainty on $\hat{\mu}$ from scan of NLL = $-2\ln\mathcal{L}$ around minimum
- **Uncertainty due to fluctuations of data**: "statistical uncertainty"
- For Gaussian pdfs: parabola, standard deviation σ follows from

$$\Delta(\text{NLL}) = \text{NLL}(\hat{\mu} \pm r \cdot \sigma) - \text{NLL}(\hat{\mu}) = r$$



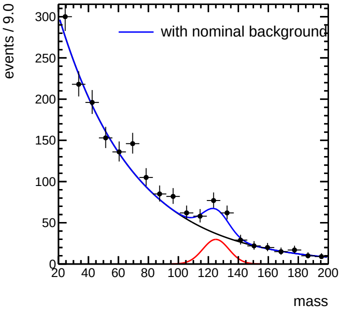
- Other cases can often be approximated by Gaussian case (and true for $n \to \infty$)
- For strongly asymmetric NLL
  - Often asymmetric intervals quoted
  - Better: variable transformation such that NLL symmetric

Uncertainty vs. Error

- **Uncertainty** reflects degree of precision with which one can deduce a parameter value from the data if one does everything correctly
  - Statistical uncertainties stem from the inherent stochastic nature of the data and finite number of observed events (can not be eliminated, only minimised)
  - Systematic uncertainties stem e. g. from the limited knowledge of the precision of the detector or approximations in theory calculations
  - Uncertainties can (in principle) be quantified
- In contrast, **errors are mistakes**
  - e. g. a lose cable or a wrong method
  - Errors can (and should) be eliminated
- In context of statistical data analysis, we mean uncertainties
  - NB: “Error bars” denote uncertainties!

Incorporation of Systematic Uncertainties

- Often, background (and signal) models subject to systematic uncertainties
  - Some (theory or experimental) parameters not exactly known, e. g. cross section or trigger efficiency
- For example, background normalisation *b* not precisely known but with some uncertainty



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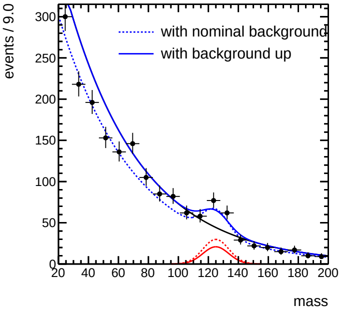

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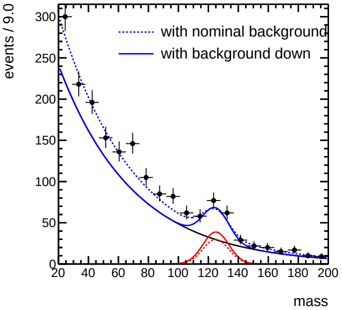
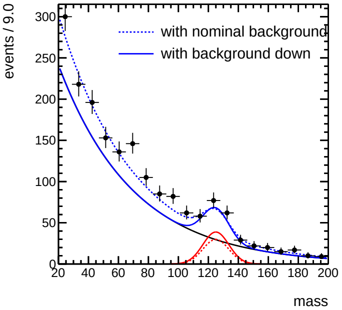


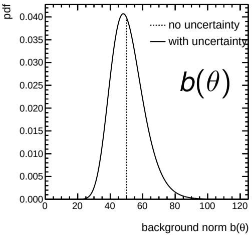
Incorporation of Systematic Uncertainties

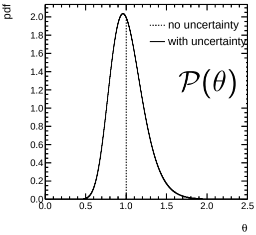
- Incorporated into likelihood via **nuisance parameters** θ

$$\mathcal{L}(\mathbf{n}_{\text{obs}};\mu,\theta) = \prod_i \mathcal{P}(n_{\text{obs}\,i};\mu,\theta)\cdot\mathcal{P}_{\tilde{\theta}}(\tilde{\theta}|\theta)$$

- Background normalisation becomes function of θ: $b \rightarrow b(\theta)$
  - θ: assumed true value, parameter of the fit
  - $\tilde{\theta}$: best knowledge, e. g. estimate from independent measurement, distributed as $\mathcal{P}_{\tilde{\theta}}(\tilde{\theta}|\theta)$







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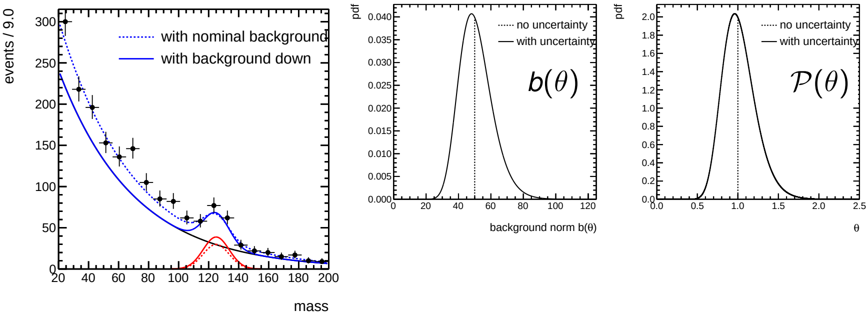
- Background normalisation becomes function of θ: $b \to b(\theta)$
- ML fit can adjust θ to a value different than $\tilde{\theta}$ to achieve better description of data but at the cost of reducing value of $\mathcal{P}_{\tilde{\theta}}(\tilde{\theta}|\theta)$



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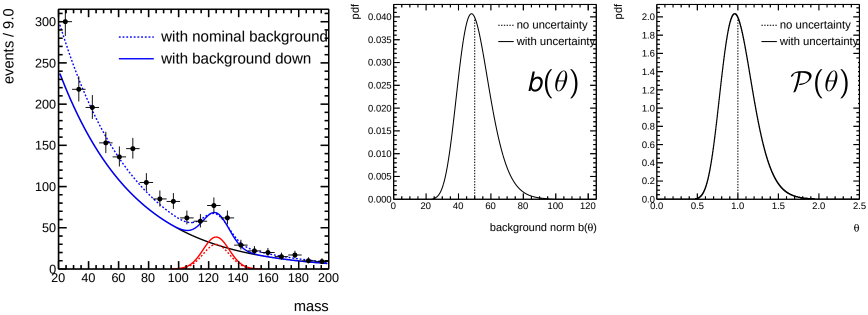
- Nomenclature
  - μ: *parameter of interest* (POI)
  - θ: *nuisance parameter* (NP)
    generally, (anti-)correlated with μ: increased uncertainty on $\hat{\mu}$



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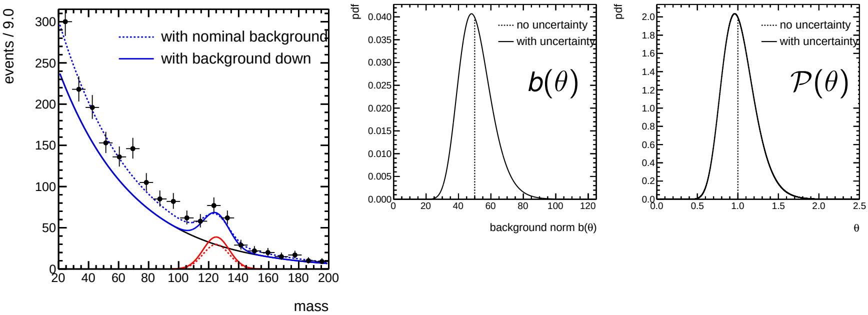

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- Nomenclature
  - μ: *parameter of interest* (POI)
  - θ: *nuisance parameter* (NP)
- $\mathcal{L}$ function of μ and θ, but only interested in μ
  $\rightarrow$ can rewrite θ as function of μ

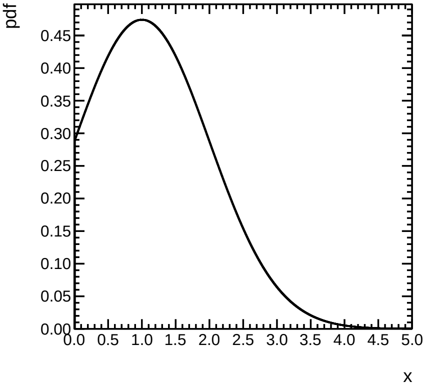
Profile likelihood $\mathcal{L}_{\text{p}}(\mu) \equiv \mathcal{L}(\mu, \hat{\theta}(\mu))$

$\hat{\theta}(\mu)$ maximises $\mathcal{L}$ for given μ (“profiled values” of θ)

- $\mathcal{L}_{\text{p}}$ in practice computationally advantageous
  - e. g. need only to consider μ instead of full parameter space (μ, θ) when computing uncertainties

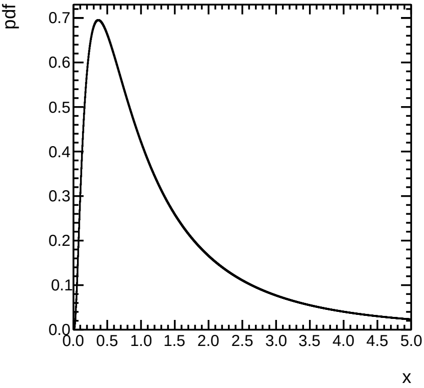
Typical Choices for $\mathcal{P}_\theta(\theta|\tilde{\theta})$

(truncated) Gaussian



$$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

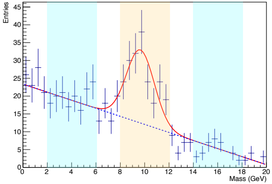
Log-normal



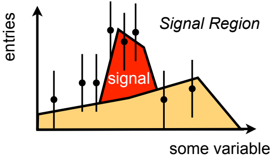
$$\frac{1}{\sqrt{2\pi}\sigma} \frac{1}{x} e^{-\frac{1}{2}(\frac{\ln(x)-\mu}{\sigma})^2}$$

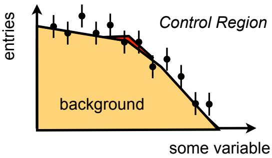
Sidebands and Control Regions

sidebands



signal/control region





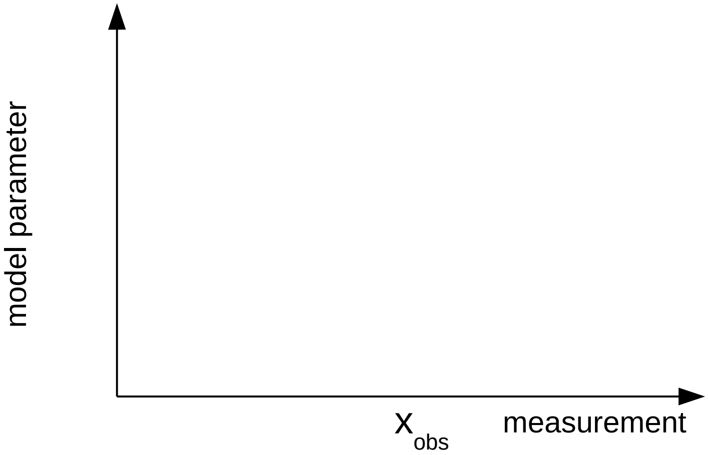
- Signal-depleted sideband/control region
  - Determine rate of background process
  - Determine shape of background processes
  - Constrain uncertainties on background prediction
- Can be incorporated into likelihood fit via **nuisance parameters**: **simultaneous determination of POI and background parameters**

Confidence Intervals

- Experiment to determine value of some parameter x measures x_{obs}
  - e. g. from maximum likelihood estimate, tag-and-probe measurement
- Want to quote "uncertainty": interval that reflects statistical precision of measurement
  - Can be estimate of standard deviation from likelihood fit
  - More generally, in particular if non-Gaussian $\mathcal{P}$: **confidence interval**
- **Confidence interval** covers on average the true value with a given probability
  - e. g. *90 % confidence level*: if experiment repeated many time, the interval covers the true value in 90 % of the cases
  - Careful: this is not the probability of the true value to lie within the interval (there is only one true value either within or not)!

Confidence Intervals

Neyman construction



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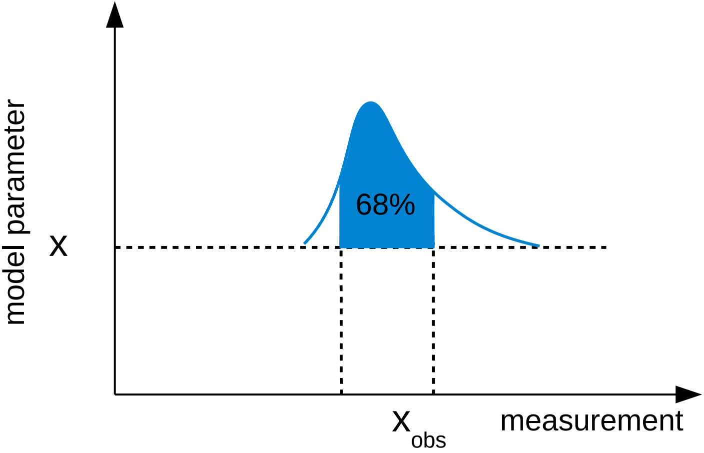

Confidence Intervals
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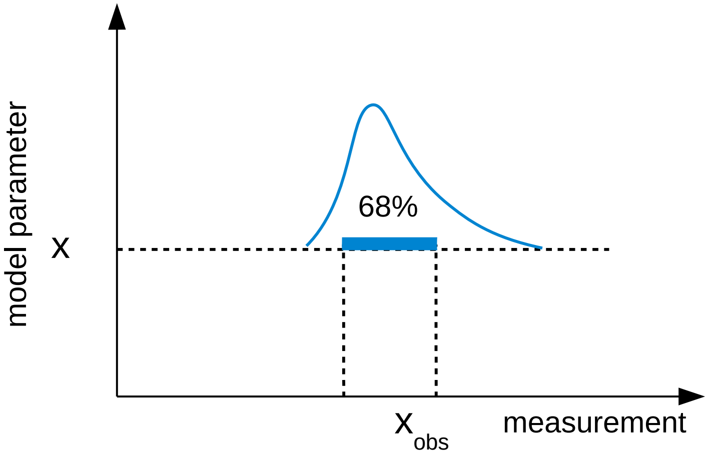

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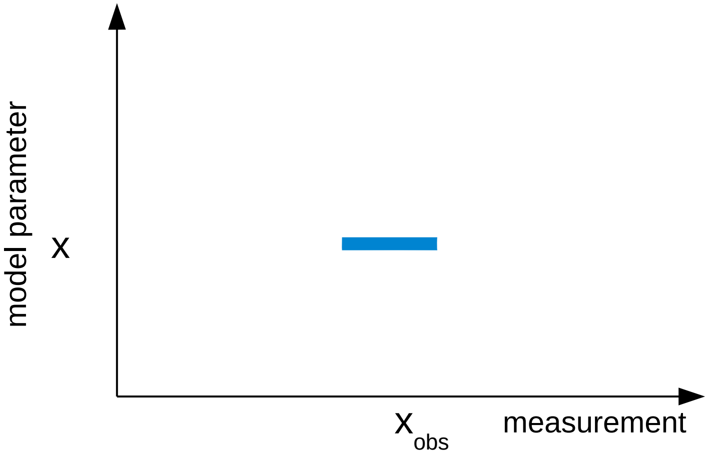

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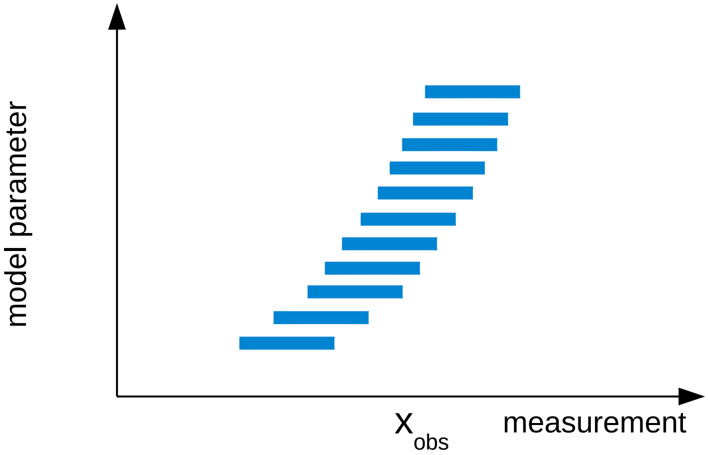

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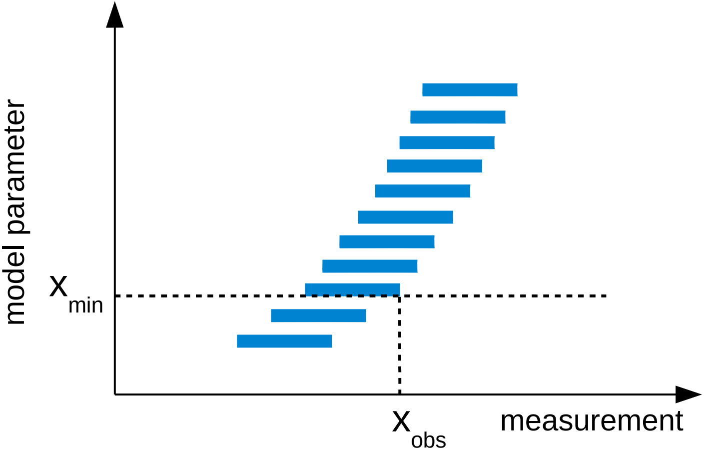

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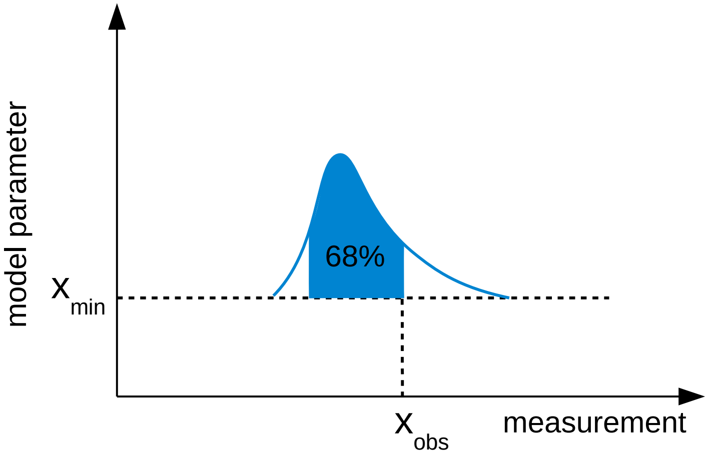

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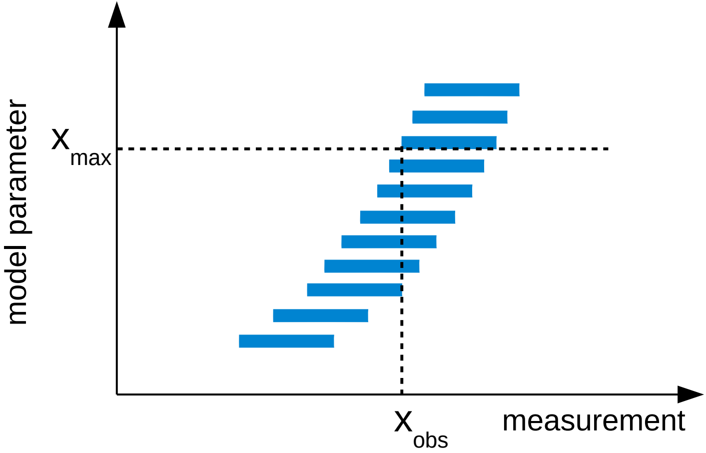

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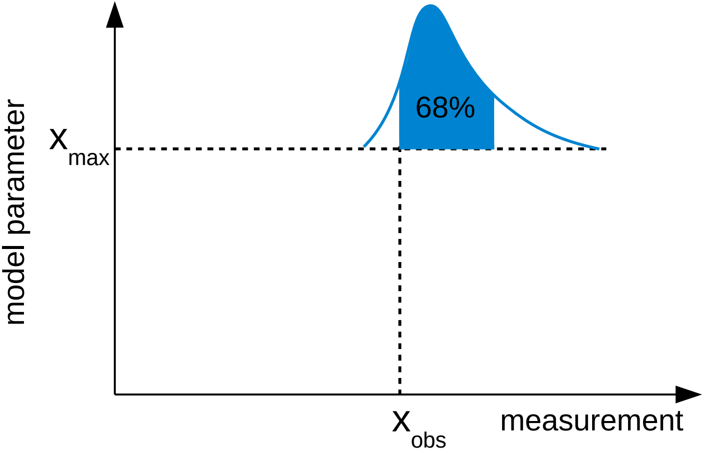

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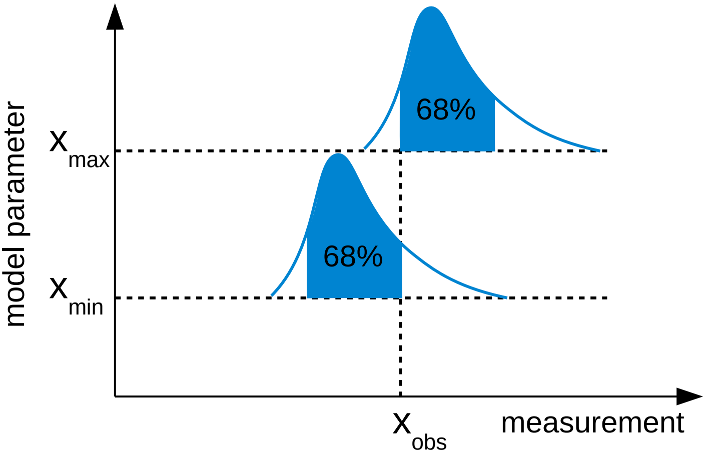
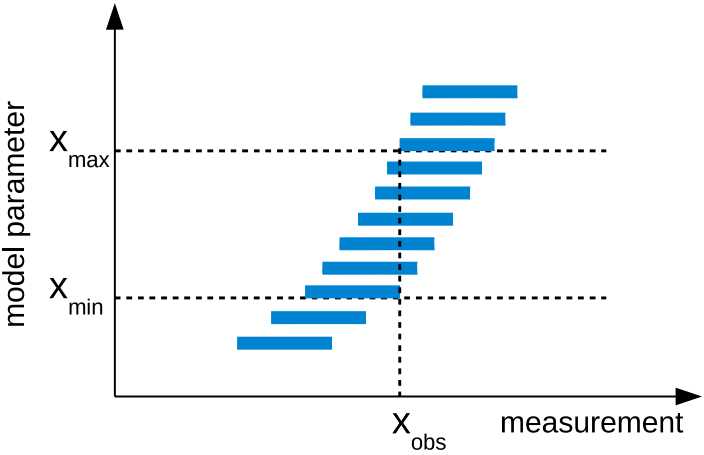


Confidence interval is $[x_{\mathrm{min}}, x_{\mathrm{max}}]$

Confidence Intervals

Neyman construction



Confidence interval is $[x_{min}, x_{max}]$

Application: Uncertainty of Efficiency

(see Exercises No. 3)

- Efficiency: **numberator = subset of denominator**
  $\rightarrow$ fully correlated, don't use Gaussian error propagation!
- Good description: **binomial uncertainties**
  - Given true efficiency ϵ
  - Draw from a population of N events: expect $\langle n \rangle = \epsilon N$ events on average
  - Variance of n

$$V[n] \equiv \sigma_n^2 = N\epsilon(1-\epsilon)$$

  - But don't know true efficiency: replace with estimator $\hat{\epsilon}$ from measurement (random sample)

$$\epsilon \rightarrow \hat{\epsilon} = \frac{n}{N}, \qquad \hat{\sigma}_n^2 = N\hat{\epsilon}(1-\hat{\epsilon})$$

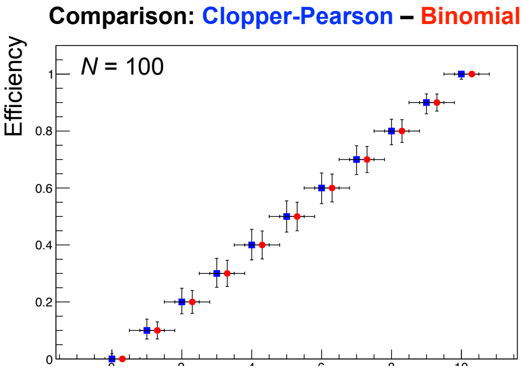
  - Uncertainty on efficiency estimator

$$\boxed{\sigma_{\hat{\epsilon}}^2 = \frac{1}{N^2}\hat{\sigma}_n^2 = \frac{\hat{\epsilon}(1-\hat{\epsilon})}{N}}$$

Application: Uncertainty of Efficiency

(see Exercises No. 3)

- Problem with binomial uncertainty from *measured* efficiency: variance $\rightarrow 0$ for $\hat{\epsilon} \rightarrow 0, 1$
- Solution: construction of proper **68 % confidence level intervals**

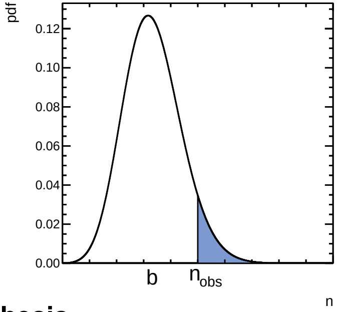


3.3.2. Hypothesis testing

p Value

- How well is the data described by my hypothesis (model)?
- For example, **how likely that observed peak in data just upward fluctuation** of the background, i. e. no Higgs boson present?
- Assume simple counting experiment
  - *b*: expected number of background events (=model)
  - n_{obs}: number of observed events
- "*p* value": **probability of upward fluctuation as large as or larger than observed in data**

$$p \equiv \mathrm{P}(n \geq n_{\text{obs}} | b) = \int_{n_{\text{obs}}}^{\infty} \mathrm{d}n \, \mathcal{P}(n|b)$$



p value is **not the probability of a hypothesis**

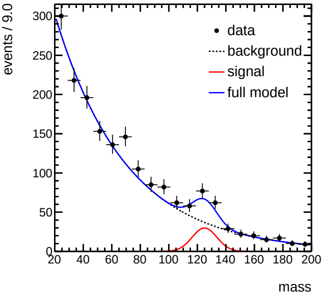
p quantifies level of (dis-)agreement between model and data:

$\rightarrow$ judgement call whether to keep model or reject it

Test Statistic

- Typically problems not as simple as a single measured quantity
  - Multiple channels/measurements, e. g. binned distribution
  - Prediction depends on several parameters, e. g. POI, nuisance params.
- Formally: use appropriate **test statistic** $t : \mathbb{R}^D \to \mathbb{R}$
  - In general any function that **combines relevant information** from an experiment, e. g. number of observed events per bin, **into one single number** reflecting the agreement between data and hypothesis
  - Likelihood is example for test statistic
- t can be used to compute p value for complex models

$$p = \int_{t_{\text{obs}}}^{\infty} \mathcal{P}(t)\,\mathrm{d}t$$



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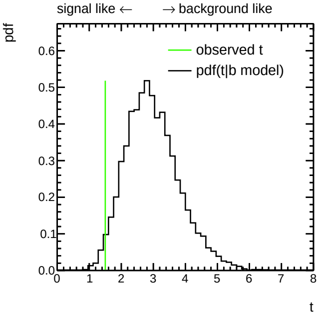
$$p = \int_{t_{\text{obs}}}^{\infty} \mathcal{P}(t)\,\mathrm{d}t$$

  - Requires pdf $\mathcal{P}(t)$ of the test statistic t: typically from simulation (*toy data*)



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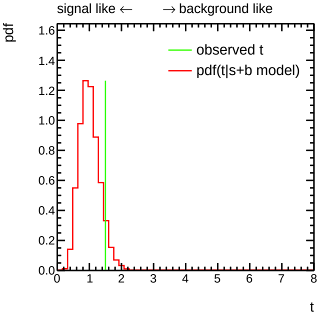
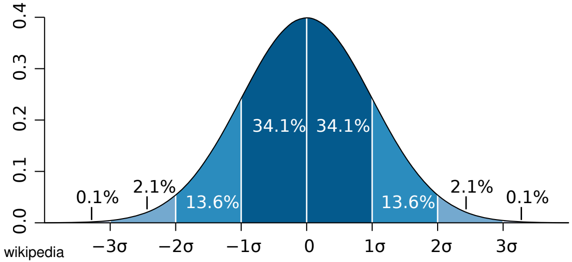


Significance

- Often, p value converted into equivalent **significance Z**:
  upward fluctuation from 0 by Z of normal-distributed variable corresponding to same p value
  - Corresponds to Z standard deviations σ of the Gaussian distribution

$$Z = \Phi^{-1}(1-p)$$

Φ: cumulative (=quantile) function of normal distribution



- **Convention** to classify effects by significance
  - 3σ: *evidence* for signal (0.3% chance of background fluctuation)
  - 5σ: *discovery* of signal (0.00006% chance of background fluctuation)

Distinguishing Hypotheses

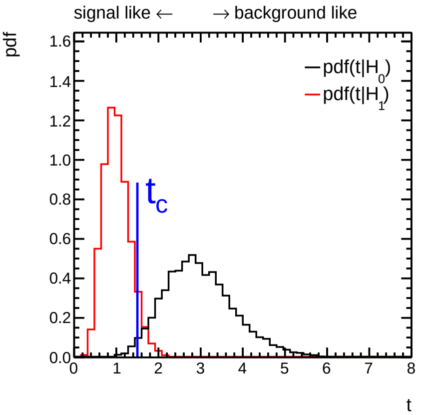
- **Cannot determine whether a hypothesis is true** (frequentist)
- But can define rules how to reject hypothesis in favour of an alternative hypothesis
- Can determine probability of wrong choice: how often wrong choice is made would the experiment be repeated very often

Distinguishing Hypotheses

- Classification problem: how to interpret outcome of an experiment?
- Want to distinguish between two alternative hypotheses, e. g.
  - H_0: there are only background events, e. g. no Higgs boson
  - H_1: there is a signal, e. g. a Higgs boson, contribution to the data

- Define **test statistic *t***, e. g. $\mathcal{L}$
- Construct pdf $\mathcal{P}(t|H_i)$ of t under H_0 and H_1
  - In reality often from simulation
- How compatible is t_{obs} with H_i?

→ Set a critical value t_c and reject H_0 in favour of H_1 if $t_{obs} < t_c$



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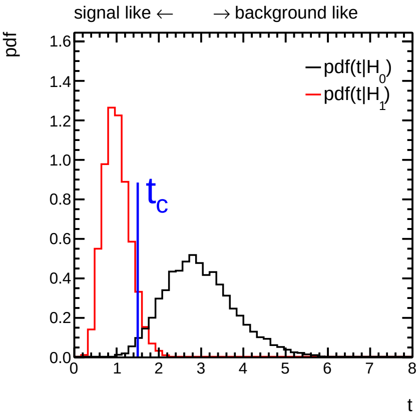
- Types of errors
  - I: reject H_0 although it is true
  - II: accept H_0 although H_1 is true

$$\text{I: } P(t < t_c | H_0) = \int_{-\infty}^{t_c} \mathrm{d}t \, \mathcal{P}(t|H_0) \equiv \alpha$$
$$\text{II: } P(t > t_c | H_1) = \int_{t_c}^{\infty} \mathrm{d}t \, \mathcal{P}(t|H_1) \equiv \beta$$

α : significance of test

$1 - \beta$: power of test



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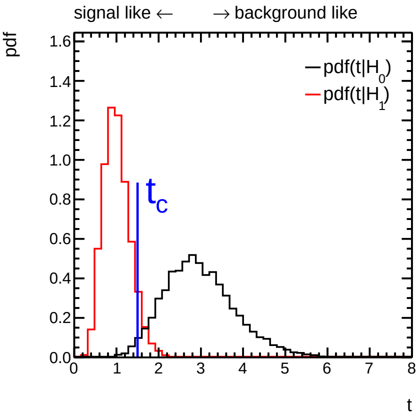
By choice of t_c (choice of α): association to one or the other hypothesis performed "**at the confidence level α**"



Difference Between α and *p* Value?

- In one sense, no difference
  - Both are $\int_x^\infty \mathrm{d}x' \, \mathcal{P}(x')$
- But **conceptually very different**
  - α is computed before one sees the data: predefined property of the test
  - *p* depends on the data (property of the data) and is a random variable

Neyman-Pearson Lemma

- When performing a test between two hypotheses (models) H_0 and H_1, the **likelihood ratio test**, which rejects H_0 in favour of H_1 if

$$\boxed{Q = \frac{\mathcal{L}_{H_1}}{\mathcal{L}_{H_0}} > Q_c} \quad \text{with } \mathrm{P}(Q > Q_c | H_0) = \alpha$$

  is the **most powerful test** at a significance level α
- The test statistic Q is called **likelihood ratio**
- For a number of reasons, usually

$$\boxed{q = -2 \ln Q}$$

Test Statistic at the LHC

- Used test statistic in particle-physics experiments evolved with time
- Hypothesis test example
  - H_0: background-only hypothesis, e. g. SM without Higgs boson
  - H_1: Higgs boson with fixed mass and signal strength μ present in data
- At the LHC commonly: profile likelihood ratio test statistics

$$q_\mu = -2 \ln \frac{\mathcal{L}(n_{\mathrm{obs}}; \mu, \hat{\hat{\theta}}(\mu))}{\mathcal{L}(n_{\mathrm{obs}}; \hat{\mu}, \hat{\theta})}$$

  - Nuisance parameters are profiled in nominator
  - Global maximum of $\mathcal{L}$ under (μ, θ) as denominator with $0 \leq \hat{\mu} \leq \mu$
  - Allows usage of certain approximations when computing $\mathcal{P}(q_\mu | H_i)$ ("asymptotic formulae" based on Wilks and Wald theorem[1])

[1] G. Cowan "Asymptotic formulae for likelihood-based tests of new physics", Eur. Phys. J. C71:1554 (2011)

3.3.3. Search for new physics (exclusion limits)

Excluding Parameters

- Assume measurement with a given sensitivity: no signal observed
- How much signal can “hide” in the bkg. fluctuations (+uncertainty)?
- **How large could a signal be at most?**

Excluding Parameters

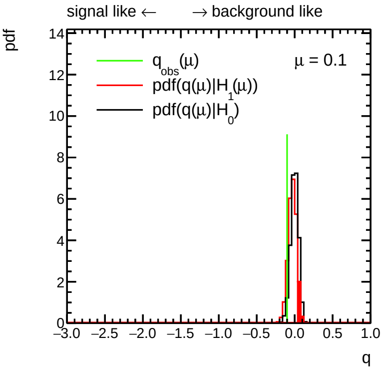
- No signal observed: **how large could a signal be at most?**
- Formally: test statistic q, depending on signal-strength modifier μ

$$q(\mu) = -2\ln\frac{\mathcal{L}_{H_1}(\mu)}{\mathcal{L}_{H_0}}$$

What is largest signal $\mu \equiv \mu_{1-\alpha}$ for which H_1 would be rejected at significance level α?

$$\alpha = \int_{q_{\mathrm{obs}}}^{\infty} \mathrm{d}q\, \mathcal{P}(q(\mu_{1-\alpha})|H_1) \equiv \mathrm{CL_{s+b}}$$

i. e. for $\mu_{1-\alpha}$: $q_{\mathrm{obs}} = q_c$



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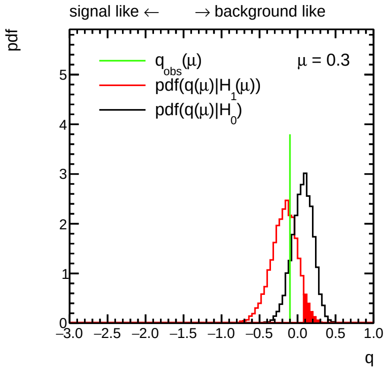

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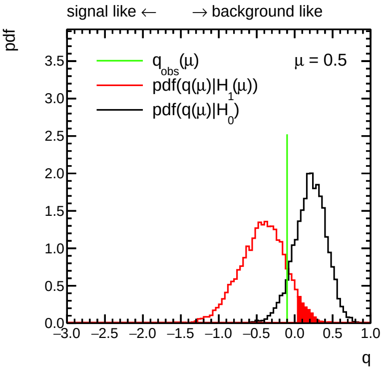

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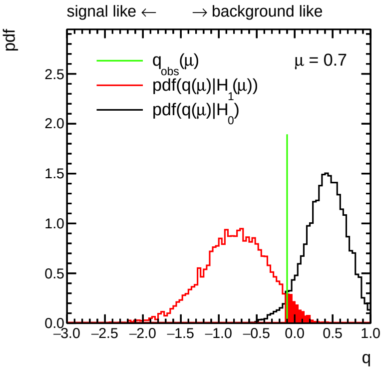

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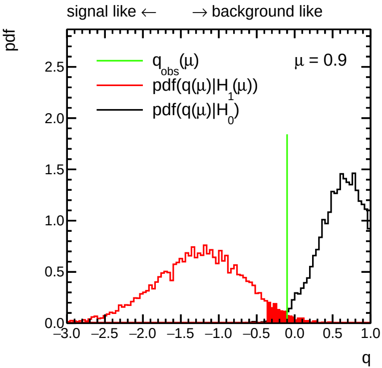

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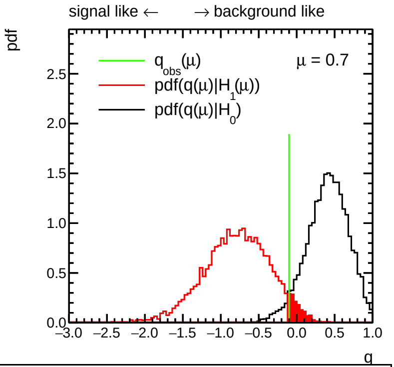


$\mu_{1-\alpha}$ is called **upper limit** on μ at confidence level C.L. $= 1 - \alpha$

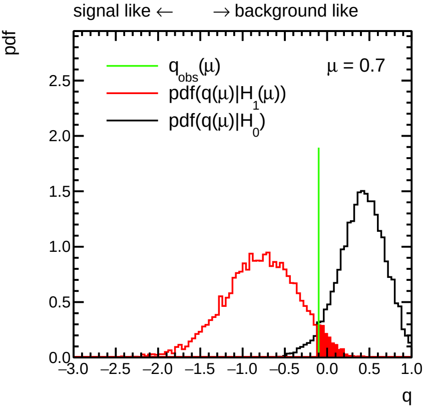
In particle physics usually 95 % C.L. limit

(Observed) Upper Limit: Interpretation

- "Maximal signal that we would still reject"
- 95 % C.L. upper limit on μ: largest value of μ that would still be rejected in a test with significance 5% given the data
  - NB: limit is a **function of the data** (depends on q_{obs})!
- μ for which $\text{CL}_{\text{s+b}} = 0.05$:

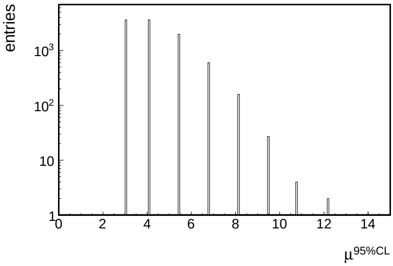
$$0.05 = \int_{q_{\text{obs}}}^{\infty} \mathrm{d}q \, \mathcal{P}(q(\mu_{95})|H_1)$$

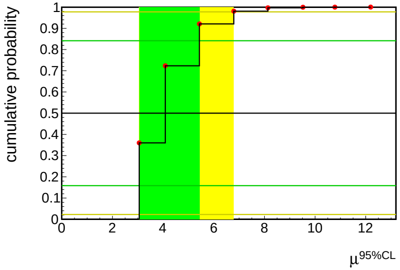
- Upper **limit covers true value ($\mu_{\text{true}} < \mu_{95}$) with probability C.L. $=$ 95 %**
  - If the experiment is repeated many times, μ_{95} would be larger than μ_{true} in 95 % of the cases
- Still **5 % chance of wrong exclusion**, i. e. that $\mu_{\text{true}} > \mu_{95}$



Expected Limit

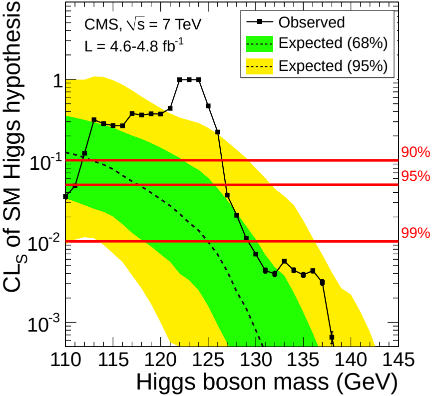
- **Estimate what observed limit would look like in case of no signal**
- Obtained e. g. from *toy dataset*
  - Sample toy data for q under background-only hypothesis from $\mathcal{P}(q|H_0)$
  - Treat each as observation and compute μ_{95} limit
  - Obtain quantiles from distribution of all μ_{95}
- **Expected limit = median of μ_{95} distribution**
  - 16 and 84% quantiles: 68% confidence interval
  - 2.5 and 97.5% quantiles: 95% confidence interval
    $\rightarrow$ “Brazilian band” plots





Before the Higgs-Boson Discovery

- Combination of Higgs-boson search results by CMS [Phys.Lett. B710 (2012) 26]



- Tested hypotheses
  - H_0: no Higgs boson
  - H_1: SM Higgs boson
- Test statistic q evaluated for SM Higgs boson of different mass ($\mu = 1$ in each case)

- Excluding a SM Higgs boson at 95% CL with masses
  - $m_H > 118$ GeV expected (from toy data under H_0)
  - $m_H > 127$ GeV observed (from real data)

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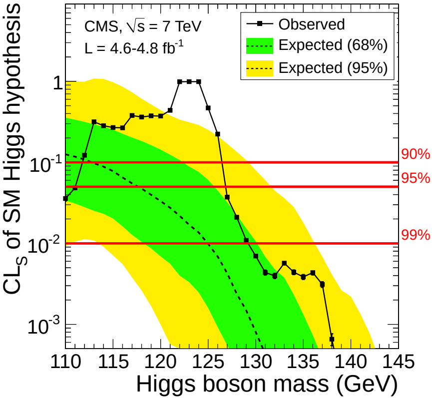

- Tested hypotheses
  - H_0: no Higgs boson
  - H_1: SM Higgs boson
- Test statistic q evaluated for SM Higgs boson of different mass ($\mu = 1$ in each case)

- Observed exclusion weaker than expected (smaller mass range)
- Around $m_H = 125\,\text{GeV}$, q_{obs} differs significantly from expectation: **indication that H_0 is wrong!**